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Structural properties of positive linear time-invariant difference-algebraic equations ☆

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Abstract

In this paper we analyze positive difference-algebraic equations. Conditions regarding some of the matrices involved in the solution of this kind of systems are described. Geometrical conditions are used to characterize positive structural properties. © 2002 Elsevier Science Inc. All rights reserved.

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1. Introduction

The characterization of positive systems attracts interest because this kind of system appears in the modeling of many processes, for instance, economic models, circuit networks, chemical and power systems. In these models state variables represent population, measure, mass, etc., and therefore, there are nonnegative.

Many aspects of positive standard systems have been considered by different authors. Coxson and Shapiro [5] and Rumchev and James [10] analyzed structural properties of these systems. Valcher [12] and Bru et al. [3] gave canonical forms of

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the reachability property. The pole assignment problem was studied by Rumchev and James [11]. Recently, Rumchev and Caccetta [9] provided a survey of reachability and controllability properties. A complete introduction to positive linear standard systems can be found in [7].

For singular systems without nonnegative restrictions, results have been described by different authors, for instance see [6,8] and the references there in. Positive singular systems have only recently studied in the control theory.

In this work, we study positive discrete-time singular systems, that is, linear time-invariant difference-algebraic equations with nonnegative solutions, deriving conditions over the coefficient matrices of the system such that the solution of the singular system has a positive behavior. In addition, we characterize positive reachability and positive controllability properties. To this end, we construct reachability cones. We will see that the results obtained are generalizations of those known for positive standard systems.

Consider the discrete-time singular system, that is, the system of linear time-invariant difference-algebraic equations (in short DAE or system)

$$\begin{aligned} Ex_{k+1} &= Ax_k + Bu_k, \\ y_k &= Cx_k + Du_k, \end{aligned} \tag{1}$$

where $E, A \in \mathbb{R}^{n\times n}$, rank $(E) \leqslant n$, $B \in \mathbb{R}^{n\times m}$, $C \in \mathbb{R}^{p\times n}$, $D \in \mathbb{R}^{p\times m}$, $k \in \mathbb{Z}_+$. System (1) is denoted by (E, A, B, C, D). In this paper, we consider a DAE given by (E, A, B), since we are interested in the study of reachability and controllability properties. Note that we can consider the system (E, A, C) if we want to analyze the dual properties. When $E = I$, system (1) is called a standard system. It is well known (see [6,8]) that if there exists a scalar $\lambda \in \mathbb{C}$ such that $\det[\lambda E - A] \neq 0$ (i.e. system (1) satisfies the regularity condition), we can obtain a state solution. A state solution to DAE (1) is given by

$$\begin{aligned} x_k = {} & (\widehat{E}^D\widehat{A})^k\widehat{E}^D\widehat{E}x_0 + \sum_{i=0}^{k-1}\widehat{E}^D(\widehat{E}^D\widehat{A})^{k-i-1}\widehat{B}u_i \\ & -(I - \widehat{E}^D\widehat{E})\sum_{i=0}^{q-1}(\widehat{E}\widehat{A}^D)^i\widehat{A}^D\widehat{B}u_{k+i}, \end{aligned}$$

where $\widehat{E} = (\lambda E - A)^{-1}E$, $\widehat{A} = (\lambda E - A)^{-1}A$, $\widehat{B} = (\lambda E - A)^{-1}B$, and x_0 is an admissible initial condition, q is the smallest nonnegative integer such that rank$(\widehat{E}^q)$ = rank$(\widehat{E}^{q+1})$, called index of $\widehat{E}$, ind $(\widehat{E})$, and M^D denotes the Drazin inverse of a matrix M (see [1]). The set of admissible initial conditions $\mathscr{X}_0$ is given by

$$\mathscr{X}_0 = \mathrm{Im}\big[\widehat{E}^D\widehat{E}, H_0, \ldots, H_{q-1}\big], \tag{2}$$

where $H_i = (I - \widehat{E}^D\widehat{E})(\widehat{E}\widehat{A}^D)^i\widehat{A}^D$, $i = 0, \ldots, q-1$. Note that matrices $\widehat{E}$ and $\widehat{A}$ satisfy $\widehat{E}\widehat{A} = \widehat{A}\widehat{E}$, which is a basic property for obtaining the above explicit solution in terms of the Drazin inverse of $\widehat{E}$ and $\widehat{A}$, respectively.

Since the solution of a DAE is given in terms of the Drazin inverse, we have provided a background to generalized inverses (see [4]). It is well known that the Drazin inverse of a matrix $A \in \mathbb{R}^{n\times n}$ is the matrix A^D satisfying: (i) $A^D A A^D = A^D$, (ii) $AA^D = A^D A$ and (iii) $A^{k+1}A^D = A^k$, where $k \geqslant \mathrm{ind}(A)$, where $\mathrm{ind}(A)$ is the smallest nonnegative integer k such that $\mathrm{rank}(A^k) = \mathrm{rank}(A^{k+1})$.

If $\mathrm{ind}(A) \leqslant 1$, then A^D also satisfies $A\,A^D A = A$. In this case the Drazin inverse of A is called the group inverse of A and it is denoted by $A^\#$. Then the group inverse can be considered as a special case of the Drazin inverse. The group inverse of A is usually defined as follows: given $A \in \mathbb{R}^{n\times n}$, if $A^\#$ exists, the group inverse is the unique matrix satisfying conditions (i), (ii) and $AA^\# A = A$.

Given a matrix partitioned as

$$M = \begin{bmatrix} A & B \\ C & D \end{bmatrix}$$

with $A \in \mathbb{R}^{r\times r}$ and $\mathrm{rank}(M) = \mathrm{rank}(A) = r$, then

$$M^D = \begin{bmatrix} I \\ CA^{-1} \end{bmatrix} \left[(AS)^2\right]^D A[I A^{-1}B],$$

where $S = I + A^{-1}BCA^{-1}$ (see [4]).

The set of nonnegative vectors of length n is denoted by $\mathbb{R}^n_+$. If $x \in \mathbb{R}^n_+$ then it is denoted by $x \geqslant 0$. A nonnegative matrix $A = [a_{ij}]$ with $a_{ij} \geqslant 0$ is denoted by $A \geqslant O$.

2. Characterization of positive DAE

In this section we characterize the positive solution of the DAE (E, A, B) by means of the coefficient matrices. First, we define positive system (positive DAE).

Definition 1. System (E, A, B) is called positive if, for every admissible initial state $x_0 \geqslant 0$ and for every nonnegative control sequence, the state trajectory belongs to $\mathbb{R}^n_+$, that is $x_k \geqslant 0\ \forall k \in \mathbb{Z}_+$, $\forall x_0 \in \mathscr{X}_0 \cap \mathbb{R}^n_+$ and $\forall u_j \geqslant 0$, $j = 0, 1, \ldots, k+q-1$.

In general, in the following we suppose that matrices E and A of the system commute, $EA = AE$.

The following algebraic characterization of positive DAE is given in [2]. The proof is given for completeness.

Proposition 1. *Consider the system (E, A, B). Suppose that $EE^D \geqslant 0$ and $EA = AE$. The system (E, A, B) is positive if and only if $E^D A \geqslant 0$, $E^D B \geqslant 0$ and $(I - E^D E)(EA^D)^i A^D B \leqslant 0$, $i = 0, 1, \ldots, q-1$, where q is the index of E.*

Proof. Suppose that the system is positive. The nonnegative trajectory of the system, when $EA = AE$, is given by

$$x_k = (E^D A)^k E^D E x_0 + \sum_{i=0}^{k-1} E^D (E^D A)^{k-i-1} B u_i$$
$$-(I - E^D E) \sum_{i=0}^{q-1} (E A^D)^i A^D B u_{k+i} \tag{3}$$

with $x_0 \in \mathscr{X}_0 \cap \mathbb{R}^n_+$. Since $EE^D \geqslant 0$ then $x_0 = EE^D e_i \geqslant 0$ is an admissible initial state, for all $i = 1, \ldots, n$. In this case, using the sequence control $u_j = 0$, $j = 0, 1, \ldots, k + q - 1$, the nonnegative state of the trajectory at time $k = 1$ is given by

$$x_1 = (E^D A) E E^D x_0 = (E^D A) E E^D E E^D e_i.$$

Using $E^D A = A E^D$ and Drazin inverse properties, we have

$$x_1 = (E^D A) E E^D e_i = A E^D e_i \geqslant 0$$

for all $i = 1, \ldots, n$. Then $E^D A \geqslant 0$.

Now, take $x_0 = 0$, $u_0 = e_i \in \mathbb{R}^n_+$, $u_j = 0$, $j = 1, \ldots, q$, where q is the index of E. At time $k = 1$, $x_1 = E^D B e_i \geqslant 0$, and by applying this process to all $i = 1, \ldots, n$, we obtain $E^D B \geqslant 0$.

Choosing $u_k = e_i$ and $u_j = 0$, $j \neq k$, $j = 1, \ldots, k + q - 1$, we have

$$x_k = -(I - E^D E) A^D B e_i \geqslant 0$$

for all $i = 1, \ldots, n$. Then $(I - E^D E) A^D B \leqslant 0$.

Taking $u_{k+h} = e_i$ for all $i = 1, \ldots, n$ and $u_j = 0$, $j \neq k + h$, $j = 1, \ldots, k + q - 1$, we obtain

$$x_k = -(I - E^D E)(E A^D)^h A^D B e_i \geqslant 0$$

for all $i = 1, \ldots, n$. Then $(I - E^D E)(E A^D)^h A^D B \leqslant 0$ for all $h = 1, \ldots, q - 1$.

Conversely, it is easy to check that if $(I - E^D E)(E A^D)^i A^D B \leqslant 0$ for all $i = 0, 1, \ldots, q - 1$, $E^D B \geqslant 0$ and $E^D A \geqslant 0$, then the trajectory (3) is nonnegative, $x_k \geqslant 0$, $\forall u_j \geqslant 0$, $j = 1, \ldots, k + q - 1$, $k \in \mathbb{Z}_+$. □

Remark 1. If the condition $EA = AE$ does not hold, the matrices E and A can be replaced by $\widehat{E}$ and $\widehat{A}$, respectively, and matrix B can be replaced by $\widehat{B}$, to obtain the same result.

If the system (E, A, B) satisfies the regularity condition, then that system is equivalent to the canonical forward–backward form (see [6]), given by

$$\overline{E}\overline{x}_{k+1} = \overline{A}\overline{x}_k + \overline{B} u_k \tag{4}$$

with

$$\overline{E} = \operatorname{diag}(I_{n_1}, N), \quad \overline{A} = \operatorname{diag}(A_1, I_{n_2}), \quad n_1 + n_2 = n, \quad \overline{B} = \begin{bmatrix} B_1 \\ B_2 \end{bmatrix},$$

and N is nilpotent. In this case $\overline{E}^D\overline{E} = \mathrm{diag}(I_{n_1}, O)$, $\overline{E}^D\overline{A} = \mathrm{diag}(A_1, O)$, $\overline{E}^D\overline{B} = [B_1^{\mathrm{T}} \;\; O]^{\mathrm{T}}$ and $(I - E^D E)(EA^D)^i A^D B = [0 \;\; (N^i B_2)^{\mathrm{T}}]^{\mathrm{T}}$. Hence, a forward–backward system is positive if and only if $A_1 \geqslant 0$, $B_1 \geqslant 0$ and $N^i B_2 \leqslant 0$, $i = 0, 1, \ldots, q-1$.

Note that the condition of nonnegativity of the matrices $E^D E$, $E^D A$, $E^D B$ is a generalization of the positive characterization of the standard case, $E = I$, that is, the matrices A and B are nonnegative. This fact leads to the study of certain properties over nonnegative matrices in order to determine the Drazin inverse of special cases of nonnegative matrices.

3. Nonnegative generalized inverses and application to positive DAE

Now, consider a nonnegative matrix $E \in \mathbb{R}_+^{n\times n}$. There exists a permutation matrix P such that

$$PEP^{\mathrm{T}} = \begin{bmatrix} M & E_{12} \\ E_{21} & E_{22} \end{bmatrix} \in \mathbb{R}_+^{n\times n}, \tag{5}$$

where $r = \mathrm{rank}(E)$, M is a nonsingular monomial matrix of order less than or equal to r. If matrix M is a nonsingular matrix, E_{22} is nilpotent and $E_{12} = E_{21} = O$, the Drazin inverse of the above matrix is given by $E^D = P^{\mathrm{T}}\mathrm{diag}(M^{-1}, O)P$. We shall now study some interesting particular cases.

Proposition 2. *Consider the matrix $E \in \mathbb{R}_+^{n\times n}$. Suppose that there exists a permutation matrix such that*

$$E = P^{\mathrm{T}} \begin{bmatrix} M_r & E_{12} \\ E_{21} & E_{22} \end{bmatrix} P \tag{6}$$

where $r = \mathrm{rank}(E)$ and M_r is a nonsingular monomial matrix of size r. We have:

1. *If $E_{22} = O$ and $E_{12}E_{21} = O$, then*

$$E^D = P^{\mathrm{T}} \begin{bmatrix} M_r^{-1} & M_r^{-2}E_{12} \\ E_{21}M_r^{-2} & E_{21}M_r^{-3}E_{12} \end{bmatrix} P \in \mathbb{R}_+^{n\times n}.$$

2. *If $E_{21} = O$, then*

$$E^D = P^{\mathrm{T}} \begin{bmatrix} M_r^{-1} & M_r^{-2}E_{12} \\ O & O \end{bmatrix} P \in \mathbb{R}_+^{n\times n}.$$

3. *If $E_{12} = O$, then*

$$E^D = P^{\mathrm{T}} \begin{bmatrix} M_r^{-1} & O \\ E_{21}M_r^{-2} & O \end{bmatrix} P \in \mathbb{R}_+^{n\times n}.$$

Proof. It is easy to prove the above proposition if $r = \mathrm{rank}(E) = \mathrm{rank}(M_r)$ then $E_{22} = E_{21}M_r^{-1}E_{12}$. □

Now we consider the special structures of matrix E analyzed in the above proposition in order to weaken the sufficient condition of Proposition 1 of positive systems. In particular, we say that matrix E satisfies:

- **Condition (a)**. If under a permutation transformation P, E is given by expression (5), where M is a nonsingular monomial matrix and E_{22} is a nilpotent matrix, $E_{12} = E_{21} = O$.
- **Condition (b)**. If under a permutation transformation P, E is given by expression (6) where $r = \text{rank}(E)$, M_r is a nonsingular monomial matrix of size r, $E_{12}E_{21} = O$, and one of the blocks E_{12}, E_{21}, E_{22} is the zero block.

Proposition 3. *Consider the system* (E, A, B). *Suppose that* $E \geqslant O$ *satisfies Condition* (a) *or Condition* (b). *If* $A \geqslant 0$ *is such that* $EA = AE$, $B \geqslant 0$ *and* $(I - E^D E)(EA^D)^i A^D \leqslant 0$, $i = 0, 1, \ldots, q-1$, *where* q *is the index of* E, *then the system* (E, A, B) *is positive*.

Proof. When E satisfies Condition (a), $E^D = P^{\mathrm{T}}\text{diag}(M^{-1}, O)P \geqslant O$ and $EE^D = P^{\mathrm{T}}\text{diag}(I, O)P \geqslant O$. Moreover, if $A \geqslant 0$, then $E^D A \geqslant 0$. Hence, by Proposition 1, the trajectory (3) is nonnegative, $x_k \geqslant 0 \; \forall u_j \geqslant 0$, $j = 1, \ldots, k+q-1$, $k \in \mathbb{Z}_+$.

When E satisfies Condition (b) then $E^D \geqslant O$ and we obtain

$$EE^D = P^{\mathrm{T}} \begin{bmatrix} I_r & M_r^{-1}E_{12} \\ E_{21}M_r^{-1} & E_{21}(M_r^{-1})^2 E_{12} \end{bmatrix} P.$$

Thus, by Proposition 1, trajectory (3) is nonnegative, $x_k \geqslant 0 \; \forall u_j \geqslant 0$, $j = 1, \ldots, k+q-1$, $k \in \mathbb{Z}_+$. $\square$

4. Reachability and controllability: geometrical conditions

In this section we analyze the structural properties of a positive system. To do end, we introduce the reachability cones of this kind of system and we characterize the reachability and controllability properties by means of these cones. First, we give the following definition.

Definition 2. Consider the system $(E, A, B) \geqslant 0$ and the set of admissible initial conditions $\mathscr{X}_0$ given in (2).

(i) The state $x \in \mathbb{R}^n_+$ is positively reachable if there exist a time $k \in \mathbb{Z}_+$ and a control sequence $u_j \geqslant 0$, $j = 0, 1, \ldots, k+q-1$ transferring the state of the system from the origin at time 0, to x at time k. The system $(E, A, B) \geqslant 0$ is positively reachable if every $x \in \mathbb{R}^n_+$ is positively reachable.

(ii) The system $(E, A, B) \geqslant 0$ is positively controllable if any final state $x_f \in \mathbb{R}^n_+$ can be reachable from any initial state $x_0 \in \mathscr{X}_0$.

(iii) In this last case, when $x_f = 0$, the system (E, A, B) is called positively null-controllable.

Now, we characterize the positive null-controllable property.

Theorem 1. *Consider the system $(E, A, B) \geqslant 0$ with $EE^D \geqslant 0$ and $EA = AE$. The system is positively null-controllable if and only if $E^D A$ is a nilpotent matrix.*

Proof. If $E^D A$ is a nilpotent matrix of the order of nilpotence l, it suffices to consider the state x_k with $k \geqslant l$ and the sequence control $u_j = 0$, $j = 0, 1, \ldots, k + q - 1$, to verify that the system is positively null-controllable.

Conversely, if the positive system (E, A, B) is positively null-controllable, for any $x_0 \in \mathscr{X}_0$, then there exists a $k \in \mathbb{Z}_+$ and a control sequence $u_j \geqslant 0$, $j = 0, 1, \ldots, k + q - 1$ such that $x_k = 0$, then

$$
\begin{aligned}
0 = {} & (E^D A)^k E^D E x_0 + \sum_{i=0}^{k-1} E^D (E^D A)^{k-i-1} B u_i \\
& -(I - E^D E) \sum_{i=0}^{q-1} (E A^D)^i A^D B u_{k+i}.
\end{aligned}
$$

Since the three terms of the above expression are positive (see Proposition 1), $(E^D A)^k E^D E x_0 = 0$ for all $x_0 \in \mathscr{X}_0$. In particular for $x_0 = E^D E e_i$, we have $0 = (E^D A)^k e_i$ for all $i = 1, \ldots, n$. Hence, there exists a time $l \in \mathbb{Z}_+$ such that $(E^D A)^l = 0$, that is $E^D A$ is a nilpotent matrix. $\square$

Next we shall introduce polyhedral cones containing the reachable states in k steps of a positive system. From these cones we shall establish geometrical characterizations of the structural properties.

Consider a positive system $(E, A, B) \geqslant 0$ with $q = \operatorname{index}(E)$, $EE^D \geqslant 0$ and $EA = AE$. The set of positively reachable states in k steps is the cone, $\mathscr{R}_k(E, A, B)$, generated by the columns of the nonnegative matrix

$$
\begin{bmatrix} E^D B \; E^D E^D A B & \cdots & E^D (E^D A)^{k-1} B \\
-(I - E^D E) A^D B & \cdots & -(I - E^D E)(E A^D)^{q-1} A^D B \end{bmatrix}.
$$

Denote by $\mathscr{R}_\infty(E, A, B) = \bigcup_{k=1}^{\infty} \mathscr{R}_k(E, A, B)$ the set of all reachable states in a finite time. Then the system $(E, A, B) \geqslant 0$ is positively reachable if and only if $\mathscr{R}_\infty(E, A, B) = \mathbb{R}^n_+$.

We consider the cones

$$
\mathscr{F}_k(E, A, B) = \langle E^D B, E^D E^D A B, \ldots, E^D (E^D A)^{k-1} B \rangle
$$

and

$$\mathscr{B}(E,A,B) = \left\langle -(I-EE^D)A^DB, \ldots, -(I-EE^D)(EA^D)^{q-1}A^DB \right\rangle,$$

and the set

$$\mathscr{F}_\infty(E,A,B) = \bigcup_{k=1}^{\infty} \mathscr{F}_k(E,A,B)\,.$$

Now, we may provide the geometrical characterization of positive reachable property. First, we recall that the direct sum of cones is denoted by $K = K_1 \oplus K_2$, and the subspace $K - K$ is denoted by $\operatorname{span}(K)$.

Theorem 2. *The positive system $(E,A,B) \geqslant 0$ with $EE^D \geqslant 0$ and $EA = AE$ is positively reachable if and only if*

$$\mathscr{F}_\infty(E,A,B) = \mathscr{P}\mathbb{R}^n_+ \quad \text{and} \quad \mathscr{B}(E,A,B) = (I-\mathscr{P})\,\mathbb{R}^n_+,$$

where

$$\mathscr{P} = \begin{bmatrix} I_{n_1} & O \\ O & O \end{bmatrix} \quad \text{and} \quad \operatorname{rank}(EE^D) = n_1.$$

Proof. By construction, $\mathscr{R}_k(E,A,B) = \mathscr{F}_k(E,A,B) + \mathscr{B}(E,A,B)$ and using $E^DEE^D = E^D$ and $EE^D = E^DE$, then $\operatorname{span}(\mathscr{F}_k(E,A,B)) \subset \operatorname{Im}(EE^D)$ and $\operatorname{span}(\mathscr{B}(E,A,B)) \subset \operatorname{Im}(I - EE^D)$. Since EE^D is a projection, $\operatorname{Im}(EE^D) \cap \operatorname{Im}(I - EE^D) = \{0\}$, then

$$\operatorname{span}(\mathscr{F}_k(E,A,B)) \cap \operatorname{span}(\mathscr{B}(E,A,B)) = \{0\}$$

and hence

$$\mathscr{R}_k(E,A,B) = \mathscr{F}_k(E,A,B) \oplus \mathscr{B}(E,A,B).$$

Further, by construction, $\mathscr{R}_k(E,A) \subseteq \mathscr{R}_{k+1}(E,A)$.

Since $(E,A,B) \geqslant 0$ is positively reachable if and only if $\mathscr{R}_\infty(E,A,B) = \mathbb{R}^n_+$, and since $\mathscr{R}_k(E,A,B) = \mathscr{F}_k(E,A,B) \oplus \mathscr{B}(E,A,B)$ and $\mathscr{R}_k(E,A) \subseteq \mathscr{R}_{k+1}(E,A)$, then $(E,A,B) \geqslant 0$ is positively reachable if and only if $\mathscr{F}_\infty(E,A,B) = \mathscr{P}\mathbb{R}^n_+$ and $\mathscr{B}(E,A,B) = (I-\mathscr{P})\mathbb{R}^n_+$, where

$$\mathscr{P} = \begin{bmatrix} I_{n_1} & O \\ O & O \end{bmatrix} \quad \text{and} \quad \operatorname{rank}(EE^D) = n_1. \qquad \square$$

From the above propositions and the characterization of positive reachability of standard positive systems in terms of the existence of a monomial submatrix of order n in the reachability matrix given in [5], the following characterization is established.

Proposition 4. *Consider the system $(E,A,B) \geqslant 0$ with $EE^D \geqslant 0$, rank $(EE^D) = n_1$ and $EA = AE$. Then $\mathscr{R}_\infty(E,A,B) = \mathbb{R}^n_+$ if and only if*

$$\begin{bmatrix} E^DB & E^DE^DAB & \cdots & E^D(E^DA)^{n-1}B \end{bmatrix}$$

has an $n \times n_1$ *monomial submatrix and*

$$\begin{bmatrix} -(I - E^D E)A^D B & \cdots & -(I - E^D E)(EA^D)^{q-1} A^D B \end{bmatrix}$$

has an $n \times (n - n_1)$ *monomial submatrix.*

Now, the following is the result for the positive controllability property.

Theorem 3. *Given* $(E, A, B) \geqslant 0$ *with* $EE^D \geqslant 0$ *and* $EA = AE$, *then the system is positively controllable if and only if* $\mathscr{R}_\infty(E, A, B) = \mathbb{R}^n_+$ *and* $E^D A$ *is a nilpotent matrix.*

Note that if the positive DAE is forward–backward in the last proposition, the nilpotence of $E^D A$ is replaced by the nilpotence of A_1.

Given a DAE without nonnegative restrictions (E, A, B), the aim now is to construct a positive system equivalent to the initial one.

Theorem 4. *Consider the system* (E, A, B) *such that the matrix* $\widetilde{E} = (\alpha E + A)^{-1} E = \operatorname{diag}(I, E_{22})$ *with* $0 < \alpha < 1$ *and* $E_{22} \geqslant 0$ *is nilpotent. If*

$$\widetilde{B} = (\alpha E + A)^{-1} B = \begin{bmatrix} \widetilde{B}_1 \\ -\widetilde{B}_2 \end{bmatrix}, \quad \widetilde{B}_i \geqslant 0, \quad i = 1, 2,$$

then

(i) *Given a system* (E, A, B), *there exists a positive system* $(\overline{E}, \overline{A}, \overline{B}) \geqslant 0$ *equivalent to* (E, A, B).

(ii) *The system* $(\overline{E}, \overline{A}, \overline{B})$ *is positively reachable if and only if* $\widetilde{B}_1$ *has an* $n_1 \times n_1$ *monomial submatrix and the block matrix*

$$\begin{bmatrix} (I - \alpha E_{22})^{-1} \widetilde{B}_2 & \cdots & ((I - \alpha E_{22})^{-1} E_{22})^{q-1} (I - \alpha E_{22})^{-1} \widetilde{B}_2 \end{bmatrix}$$

has an $(n - n_1) \times (n - n_1)$ *monomial submatrix.*

Proof. (i) Using the matrices $\widetilde{E} = (\alpha E + A)^{-1} E = \operatorname{diag}(I, E_{22})$, we construct

$$\widetilde{A} = I - \alpha \widetilde{E} = (\alpha E + A)^{-1} A = \operatorname{diag}((1 - \alpha)\mathrm{I}, \mathrm{I} - \alpha \mathrm{E}_{22})$$

and define $Q = \operatorname{diag}(I, (I - \alpha E_{22})^{-1})(\alpha E + A)^{-1}$. Note that as $0 < \alpha < 1$, $(I - \alpha E_{22})$ is an M-matrix and so $(I - \alpha E_{22})^{-1} \geqslant 0$. Then we can obtain the equivalent system $(\overline{E}, \overline{A}, \overline{B}) = (QE, QA, QB)$, where

$$\overline{E} = \operatorname{diag}(I, (I - \alpha E_{22})^{-1} E_{22}), \quad \overline{A} = \operatorname{diag}((1 - \alpha)I, I)$$
$$\overline{B} = \begin{bmatrix} \widetilde{B}_1 \\ -(I - \alpha E_{22})^{-1} \widetilde{B}_2 \end{bmatrix}.$$

This system is a positive system since $\overline{E}\overline{E}^D = \operatorname{diag}(I, O) \geqslant O$, $\overline{E}\overline{A} = \overline{A}\overline{E}$, $\overline{E}^D \overline{A} = \operatorname{diag}((1 - \alpha)I, O) \geqslant O$, and

$$\overline{E}^D\overline{B} = \begin{bmatrix} \widetilde{B}_1 \\ O \end{bmatrix} \geqslant O,$$

$$(I - E^D E)(EA^D)^i A^D B = \begin{bmatrix} O \\ -\left((I - \alpha E_{22})^{-1} E_{22}\right)^i (I - \alpha E_{22})^{-1} \widetilde{B}_2 \end{bmatrix} \leqslant O$$

for all $i = 0, 1, \ldots, q-1$ with $q = \operatorname{index}(\overline{E})$.

(ii) By construction of the positive system $(\overline{E}, \overline{A}, \overline{B})$, the polyhedral cones in k steps $\mathscr{F}_k(\overline{E}, \overline{A}, \overline{B})$ and $\mathscr{B}(\overline{E}, \overline{A}, \overline{B})$ are generated by the columns of the matrices

$$\begin{bmatrix} \widetilde{B}_1 & (1-\alpha)\widetilde{B}_1 & \cdots & (1-\alpha)^{k-1}\widetilde{B}_1 \\ O & O & \cdots & O \end{bmatrix}$$

and

$$\begin{bmatrix} O & \cdots & O \\ -(I - \alpha E_{22})^{-1}\widetilde{B}_2 & \cdots & -\left((I - \alpha E_{22})^{-1} E_{22}\right)^{q-1} (I - \alpha E_{22})^{-1}\widetilde{B}_2 \end{bmatrix},$$

respectively. Thus, $(\overline{E}, \overline{A}, \overline{B})$ is positively reachable if and only if $\widetilde{B}_1$ has an $n_1 \times n_1$ monomial submatrix and

$$\left[(I - \alpha E_{22})^{-1}\,\widetilde{B}_2 \quad \cdots \quad \left((I - \alpha E_{22})^{-1} E_{22}\right)^{q-1} (I - \alpha E_{22})^{-1}\widetilde{B}_2\right]$$

has an $(n - n_1) \times (n - n_1)$ monomial submatrix. □

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