On the Reachability Index of Positive 2-D Systems

Esteban Bailo, Rafael Bru, Josep Gelonch and Sergio Romero

***Abstract*— The structural property of local reachability for positive 2-D systems refers to single local states. The smallest number of steps needed to reach all local states of a system is the local reachability index of the system. This index may exceed the system dimension. Some authors have studied upper bounds on the local reachability index for specific positive 2-D systems and have suggested different upper bounds for any positive 2-D system. In this paper, the local reachability index for a special class of positive 2-D systems is characterized and an upper bound for this index is derived. A comparison with previous results is presented.**

***Index Terms*— Positive two dimensional (2-D) systems, reachability, Hurwitz products, influence digraph, local reachability index, nonnegative matrices.**

I. Introduction

During the last decade, the theory of the positive two-dimensional systems has been considerably enhanced by the study of different physical problems (see [1], [2] and [3]).

One of the most frequent representations of positive 2-D systems is the Fornasini-Marchesini model (see [4] and [1]) which is as follows:

$$x_{i+1,j+1} = A_1 x_{i+1,j} + A_2 x_{i,j+1} + B_1 u_{i+1,j} + B_2 u_{i,j+1} \qquad (1)$$

where local states $x(\cdot,\cdot) \in \mathbb{R}_+^n$, inputs $u(\cdot,\cdot) \in \mathbb{R}_+^m$, $A_1, A_2 \in \mathbb{R}_+^{n\times n}$, $B_1, B_2 \in \mathbb{R}_+^{n\times m}$ and initial global state $\chi_0 := \{x(h,k) \ : \ (h,k) \in \mathscr{C}_0\}$ being $\mathscr{C}_0 := \{(h,k) \ : \ h,k \in \mathbb{Z}, h+k = 0\}$ the separation set. Let us denote this system by (A_1, A_2, B_1, B_2).

The smallest number of steps needed to reach all local states of the system is the local reachability index of that system. The reachability index in positive 1-D systems, which is always bounded by n (see [5], [6]), has been studied in [7], [8], [9], [10] and [11]. However, on characterizing the reachability index seems to be a hard task for a positive 2-D system.

In [12], the authors suggested $\frac{n^2}{4}$ as an upper bound for the local reachability index of every positive 2-D system. Later on, in [1], the same authors reviewed the aforementioned conjecture suggesting $\frac{(n+1)^2}{4}$ as a new upper bound.

Before [1] was published, Kaczorek (see [13]) reviewed [12] and checked that the upper bound $\frac{n^2}{4}$ fails with an example. Moreover, in that same paper, the author stated that $2(n+1)$ is an upper bound for the local reachability index of the nth order positive 2-D general models. Hence, such an upper bound is also useful for Fornasini-Marchesini Models (see [4]) since these kinds of systems are a particular case of nth order positive 2-D general models. However, since $\frac{(n+1)^2}{4}$ is greater than $2(n+1)$ for all $n \geq 8$, if the first bound fails, the second one necessarily does.

The paper has been organized as follows: Section II introduces some notations and basic definitions used in the paper. In Section III, the local reachability index for a special class of positive 2-D systems is completely characterized. Moreover, for this class of systems is checked that the corresponding indices are always bounded by $\frac{(n+1)^2}{4}$, they even turn to be $\frac{(n+1)^2}{4}$ in suitable conditions.

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II. Notations and Preliminary Definitions

Denote by $|i|_n = 1 + ((i-1) \bmod n)$, $i \in \mathbb{N}$; by $\lfloor z \rfloor$ the lower integer-part of $z \in \mathbb{R}$ and by $col_j(A)$ the jth column of the matrix A.

Definition 1: Hurwitz products of the $n \times n$ matrices, A_1 and A_2 are defined as

$$\begin{array}{l} A_1{}^{i}\sqcup\!\sqcup^{j} A_2 = 0, \qquad \text{when either } i \text{ or } j \text{ is negative,} \\ A_1{}^{i}\sqcup\!\sqcup^{0} A_2 = A_1^{j}, \quad \text{if } i \geq 0, \qquad A_1{}^{0}\sqcup\!\sqcup^{j} A_2 = A_2^{j}, \quad \text{if } j \geq 0, \\ A_1{}^{i}\sqcup\!\sqcup^{j} A_2 = A_1(A_1{}^{i-1}\sqcup\!\sqcup^{j} A_2) + A_2(A_1{}^{i}\sqcup\!\sqcup^{j-1} A_2), \quad \text{if } i,j > 0. \end{array}$$

Note that $\sum_{i+j=l} A_1{}^{i}\sqcup\!\sqcup^{j} A_2 = (A_1 + A_2)^{\ell}$.

Definition 2: (see [1]) A 2-D state-space model (1) is **(positively) locally reachable** if, upon assuming $\chi_0 = 0$, for every $x^* \in \mathbb{R}_+^n$, there exists $(h,k) \in \mathbb{Z}\times\mathbb{Z}$, $h+k > 0$, and a nonnegative input sequence $u(\cdot,\cdot)$ such that $x(h,k) = x^*$. When so, the state is said to be **(positively) reachable in** $h+k$ **steps**. The smallest number of steps that allows to reach every nonnegative local state represents **the local reachability index** I_{LR} of such system.

Local reachability is equivalent to the possibility of reaching every vector of the standard basis of $\mathbb{R}^n$ or equivalently to a corresponding positive monomial vector, that is, any positive multiples of the ith vector of the standard basis of $\mathbb{R}^n$. Denote by y_i any ith positive monomial vectors. In the same way, a monomial matrix is a nonsingular matrix having a unique positive entry in each row and column.

Therefore, the study of local reachability can be reduced to the analysis of the reachability matrix in k-steps (see [1])

$$\begin{aligned} \mathscr{R}_k &= [B_1 \;\; B_2 \;\; A_1B_1 \;\; A_1B_2 + A_2B_1 \;\; A_2B_2 \;\; A_1^2B_1 \;\; \cdots \;\; A_2^{k-1}B_2] \\ &= [(A_1{}^{i-1}\sqcup\!\sqcup^{j} A_2)B_1 + (A_1{}^{i}\sqcup\!\sqcup^{j-1} A_2)B_2]_{i,j\geq 0,\ 0<i+j\leq k} \end{aligned}$$

when k varies over $\mathbb{N}$ since the local reachability property is held if and only if there exist n pairs $(h_i,k_i) \in \mathbb{N}\times\mathbb{N}$, $i = 1,\ldots,n$, and n indices $j = j(i) \in \{1,2,\ldots,m\}$ such that $(A_1{}^{h_i-1}\sqcup\!\sqcup^{k_i} A_2)col_j(B_1) + (A_1{}^{h_i}\sqcup\!\sqcup^{k_i-1} A_2)col_j(B_2)$ is an ith monomial vector, that is, there exists $k \in \mathbb{N}$ such that $\mathscr{R}_k$ contains an $n \times n$ monomial matrix.

Definition 3: (see [1]) Associated with system (1), a directed digraph called **2-D influence digraph** is defined. It is denoted by $\mathscr{D}^{(2)}(A_1,A_2,B_1,B_2)$, which is given by $(S,V,\mathscr{A}_1,\mathscr{A}_2,\mathscr{B}_1,\mathscr{B}_2)$, where $S = \{s_1,s_2,\ldots,s_m\}$ is the set of **sources**, $V = \{v_1,v_2,\ldots,v_n\}$ is the set of **vertices**, $\mathscr{A}_1$ and $\mathscr{A}_2$ are subsets of $V\times V$ whose elements are called $\mathscr{A}_1$**-arcs** and $\mathscr{A}_2$**-arcs** respectively, while $\mathscr{B}_1$ and $\mathscr{B}_2$ are subsets of $S\times V$ whose elements are called $\mathscr{B}_1$**-arcs** and $\mathscr{B}_2$**-arcs** respectively. There is an $\mathscr{A}_1$-arc ($\mathscr{A}_2$-arc) from v_j to v_i if and only if the (i,j)th entry of A_1 (A_2) is nonzero. There is a $\mathscr{B}_1$-arc ($\mathscr{B}_2$-arc) from s_j to v_i if and only if the (i,j)th entry of B_1 (B_2) is nonzero.

A path $\mathscr{P}$ in $\mathscr{D}^{(2)}(A_1,A_2,B_1,B_2)$ from v_i to v_j is a sequence of adjacent arcs (i.e. $(v_i,v_{i_1}),(v_{i_1},v_{i_2}),\ldots,(v_{i_r},v_j)$). In particular, an s_j-path is a path originating from the source s_j.

Denote by p (q) the number of 1-arcs (2-arcs) occurring in a path $\mathscr{P}$. The pair (p,q) is called the **composition** of $\mathscr{P}$ and $p+q$ its **length**.

A **circuit** is defined to be a path whose extreme vertices coincide and if each vertex appears exactly once as the first vertex of an arc, the circuit is said to be a **cycle**.

A vertex v_i is called **reachable** in $p+q$ steps (briefly reachable) if the corresponding ith monomial vector is reachable in $p+q$ steps (y_i appears in the reachability matrix in $(p+q)$-steps). That is (see [1]), from the combinatorial point of view, there exist $p,q \in \mathbb{Z}_+$, $0 < p+q$, such that all s-paths of composition (p,q) finish in the same vertex v_i $((s,v_{i_1}),\ldots,(v_{i_r},v_i))$ and from the matrix point of view, $(A_1{}^{p-1}\sqcup\!\sqcup^{q} A_2)col_jB_1 + (A_1{}^{p}\sqcup\!\sqcup^{q-1} A_2)col_jB_2 = y_i$, for some $j \in \{1,\ldots,m\}$. Denote by $I_R(v_i)$ the minimum length of the s-paths reaching v_i.

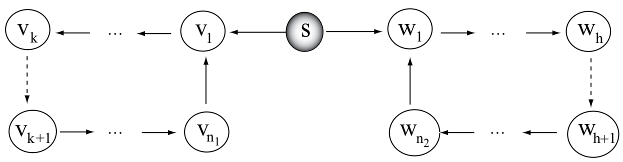

Fig. 1. Digraph for the special class of systems in section III.

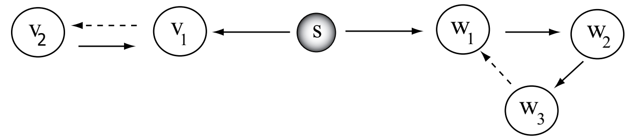

Fig. 2. Digraph for system in example 1.

III. Local Reachability Index for a special class of systems

Let us consider an nth order positive 2-D system (A_1,A_2,B_1,B_2) with $B_2=0$, B_1 a column vector with just two positive entries and 2-D influence digraph of this system consists of two loops corresponding to that unique source s with n_1 $(v_1,\dots,v_{n_1})$ and n_2 $(w_1,\dots,w_{n_2})$ vertices respectively and with all 1-arcs except for 2-arcs (v_k,v_{k+1}) and (w_h,w_{h+1}). That is, (A_1,A_2,B_1,B_2) is similar under permutation matrices to $(\hat{A}_1,\hat{A}_2,\hat{B}_1,\hat{B}_2)$ being

$$\hat{A}_1=\begin{bmatrix}\hat{A}_1' & 0\\ 0 & \hat{A}_1''\end{bmatrix},\ \hat{A}_2=\begin{bmatrix}\hat{A}_2' & 0\\ 0 & \hat{A}_2''\end{bmatrix},\ \hat{B}_1=[y_1+y_{n_1+1}] \text{ and } \hat{B}_2=0,$$

where both $\hat{A}_1'\in\mathbb{R}_+^{n_1\times n_1}$ and $\hat{A}_1''\in\mathbb{R}_+^{n_2\times n_2}$ have the following structure

$$\left[\begin{array}{cccc|cccc} 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & + \\ + & 0 & & 0 & 0 & 0 & \cdots & 0 \\ \vdots & \ddots & \ddots & \vdots & \vdots & \vdots & & \vdots \\ 0 & \cdots & + & 0 & 0 & 0 & \cdots & 0 \\ \hline 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 & + & 0 & & 0 \\ \vdots & \vdots & & \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 & 0 & \cdots & + & 0 \end{array}\right] \tag{2}$$

except for the case when both $\hat{A}_1'$ and $\hat{A}_1''$ have the structure of the upper left block of (2) and both $\hat{A}_2'\in\mathbb{R}_+^{n_1\times n_1}$ and $\hat{A}_2''\in\mathbb{R}_+^{n_2\times n_2}$ have the following structure

$$\left[\begin{array}{cccc|cccc} 0 & \cdots & \cdots & 0 & 0 & \cdots & \cdots & 0 \\ \vdots & & & \vdots & \vdots & & & \vdots \\ 0 & \cdots & \cdots & 0 & 0 & \cdots & \cdots & 0 \\ \hline 0 & \cdots & \cdots & + & 0 & \cdots & \cdots & 0 \\ \vdots & & & \vdots & \vdots & & & \vdots \\ 0 & \cdots & \cdots & 0 & 0 & \cdots & \cdots & 0 \end{array}\right] \tag{3}$$

except for the case when both $\hat{A}_2'$ and $\hat{A}_2''$ have the structure of the lower left block of (3), $+$ denoting a strictly positive entry. Moreover, 2-D influence digraph is illustrated in Fig. 1 where continuous (dotted) arrows represent 1-arcs (2-arcs).

For this class of systems we concisely denote 2-influence digraph by means of the quadruple $(n_1,\{k\};n_2,\{h\})$ with k (h) indicating where the single 2-arc of the first (second) cycle is located. In addition, n_1 and n_2 indicating the number of vertices of each cycle respectively and $n=n_1+n_2$. Finally, the vertices v_{n_1+1}, $v_{n_1+2},\dots,v_{n_1+n_2}$ are relabelled as w_1, $w_2,\dots,w_{n_2}$, to distinguish the vertices of both cycles.

Example 1: The positive 2-D system described by the matrices

$$(A_1,A_2,B_1,B_2)=$$

$$\left(\left[\begin{array}{cc|ccc} 0 & 1 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 0\\ \hline 0 & 0 & 0 & 0 & 0\\ 0 & 0 & 1 & 0 & 0\\ 0 & 0 & 0 & 1 & 0 \end{array}\right],\left[\begin{array}{cc|ccc} 0 & 0 & 0 & 0 & 0\\ 1 & 0 & 0 & 0 & 0\\ \hline 0 & 0 & 0 & 0 & 1\\ 0 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 0 \end{array}\right],\left[\begin{array}{c} 1\\ 0\\ \hline 1\\ 0\\ 0 \end{array}\right],\left[\begin{array}{c} 0\\ 0\\ \hline 0\\ 0\\ 0 \end{array}\right]\right)$$

has a 2-D influence digraph $(2,\{1\};3,\{3\})$, with $n_1=2$, $n_2=3$, $k=1$ and $h=3$, given in Fig. 2.

Firstly, let us give a necessary condition for local reachability.

Lemma 1: Let a locally reachable system with 2-D influence digraph $(n_1,\{k\};n_2,\{h\})$ then the lengths of the cycles are different; i.e., $n_1\neq n_2$.

Proof: Note that every s-path $\mathscr{P}$ finishing in the vertex v_1 has a composition $(1+r(n_1-1),r)$ for some $r\in\mathbb{Z}_+$. Furthermore, if $n_1=n_2$, it is clear that there exists another s-path with the same composition of $\mathscr{P}$ finishing in the vertex w_1. Hence, the vertex v_1 is not reachable. ■

Secondly, let us verify, in a constructive way, that a system with 2-D influence digraph $(n_1,\{k\};n_2,\{h\})$ having $n_1\neq n_2$ is locally reachable. At the same time, we will obtain the local reachability index upon assuming without a loss of generality that $n_1<n_2$. Furthermore, this index I_{LR} will be deduced in accordance to the order of c_1 and c_2 being $c_1:=\max\{k,h\}$ and $c_2:=\min\{k+n_1,h+n_2\}$.

Note that

$$k<k+n_1,\ k\le n_1<n_2<h+n_2 \text{ and } h<h+n_2. \tag{4}$$

$\boxed{\textit{First case: } c_1\ge c_2}$

In this case, considering both $c_1\ge c_2$ and (4), one finds that $c_1=h$, $c_2=k+n_1$ and $k<k+n_1\le h<h+n_2$.

Let us study the reachable vertices v_i and their corresponding indices $I_R(v_i)$. The reachable vertices will be deduced studying all the possible s-paths with the same composition in 2-D influence digraph, that is, calculating step by step the different products of Hurwitz's appearing in the reachability matrices, $(A_1{}^{i-1}\sqcup\!\sqcup^0A_2)B_1$, $(A_1{}^{i-1}\sqcup\!\sqcup^1A_2)B_1$, and so on.

To facilitate the reading, let us use a schematic representation. On the left side, let us indicate the s-paths of a given composition, more specifically let us point out the cycle chosen to construct the s-path and those arcs involved. On the other side, let us indicate the Hurwitz products pertaining to the s-path on the left side. To simplify, we will denote by y_{v_i} and y_{w_j} the monomial vectors y_i and y_{n_1+j}, respectively.

If $0<\ell\le k$, no vertices are reached by s-paths of composition $(\ell,0)$ (see Fig. 1) since

$s-$path of Composition $(\ell,0)$	Corresponding Hurwitz Products
In $1-$cycle	
$(s,v_1),(v_1,v_2),\dots,(v_{\ell-1},v_\ell)$	$(A_1{}^{\ell-1}\sqcup\!\sqcup^0A_2)y_{v_1}=A_1^{\ell-1}y_{v_1}=y_{v_\ell}$
In $2-$cycle	
$(s,w_1),(w_1,w_2),\dots,(w_{\ell-1},w_\ell)$	$(A_1{}^{\ell-1}\sqcup\!\sqcup^0A_2)y_{w_1}=A_1^{\ell-1}y_{w_1}=y_{w_\ell}$

(5)

$$\text{then}\quad (A_1{}^{\ell-1}\sqcup\!\sqcup^0A_2)B_1=(A_1{}^{\ell-1}\sqcup\!\sqcup^0A_2)(y_{v_1}+y_{w_1})=y_{v_\ell}+y_{w_\ell}.$$

However, if $k<\ell\le h$, the vertex w_ℓ is reachable since

$s-$path of Composition $(\ell,0)$	Hurwitz Products
In $1-$cycle	
No paths (Need one $2-$arc)	$(A_1{}^{\ell-1}\sqcup\!\sqcup^0A_2)y_{v_1}=A_1^{\ell-1}y_{v_1}=0$
In $2-$cycle	
$(s,w_1),(w_1,w_2),\dots,(w_{\ell-1},w_\ell)$	$(A_1{}^{\ell-1}\sqcup\!\sqcup^0A_2)y_{w_1}=A_1^{\ell-1}y_{w_1}=y_{w_\ell}$

(6)

$$\text{then}\quad (A_1{}^{\ell-1}\sqcup\!\sqcup^0A_2)B_1=(A_1{}^{\ell-1}\sqcup\!\sqcup^0A_2)(y_{v_1}+y_{w_1})=y_{w_\ell}.$$

Note that $I_R(w_\ell) = \ell$ if $k < \ell \leq h$. Moreover, no other vertices are reached by s-paths of composition $(\ell, 0)$ if $h < \ell$ since $({A_1}^{\ell-1} \sqcup\sqcup^0 A_2) B_1 = 0$.

Likewise, using $c_2 = k + n_1 \leq h = c_1$ and knowing that if $i = \ell$, $\forall\ \ell \geq 0$, Hurwitz products are

$$({A_1}^{\ell-1} \sqcup\sqcup^1 A_2) B_1 = (A_1^{\ell-1} A_2 + A_1^{\ell-2} A_2 A_1 + \cdots + A_1 A_2 A_1^{\ell-2} + A_2 A_1^{\ell-1}) B_1;$$

then,

if $0 < \ell \leq k-1$:

$s-$path of Composition $(\ell,1)$	Hurwitz Products
In $1-$cycle	
No paths (Need k $1-$arcs)	$({A_1}^{\ell-1} \sqcup\sqcup^1 A_2) y_{v_1} = 0$
In $2-$cycle	
No paths (Need h $1-$arcs)	$({A_1}^{\ell-1} \sqcup\sqcup^1 A_2) y_{w_1} = 0$

if $k \leq \ell < k + n_1$:

$s-$path of Composition $(\ell,1)$	Hurwitz Products
In $1-$cycle	
$(s, v_1), \ldots, (v_k, v_{\vert k+1 \vert_{n_1}}), \ldots, (v_{\vert \ell \vert_{n_1}}, v_{\vert \ell+1 \vert_{n_1}})$	$({A_1}^{\ell-1} \sqcup\sqcup^1 A_2) y_{v_1} = y_{v_{\vert \ell+1 \vert_{n_1}}}$
In $2-$cycle	
No paths (Need h $1-$arcs)	$({A_1}^{\ell-1} \sqcup\sqcup^1 A_2) y_{w_1} = 0$

if $k + n_1 \leq \ell < h$:

$s-$path of Composition $(\ell,1)$	Hurwitz Products
In $1-$cycle	
No paths (Need two $2-$arcs)	$({A_1}^{\ell-1} \sqcup\sqcup^1 A_2) y_{v_1} = 0$
In $2-$cycle	
No paths (Need h $1-$arcs)	$({A_1}^{\ell-1} \sqcup\sqcup^1 A_2) y_{w_1} = 0$

if $h \leq \ell < k + n_2 - 1$:

$s-$path of Composition $(\ell,1)$	Hurwitz Products
In $1-$cycle	
No paths (Need two $2-$arcs)	$({A_1}^{\ell-1} \sqcup\sqcup^1 A_2) y_{v_1} = 0$
In $2-$cycle	
$(s, w_1), \ldots, (w_h, w_{\vert h+1 \vert_{n_2}}) \ldots, (w_{\vert \ell \vert_{n_2}}, w_{\vert \ell+1 \vert_{n_2}})$	$({A_1}^{\ell-1} \sqcup\sqcup^1 A_2) y_{w_1} = y_{w_{\vert \ell+1 \vert_{n_2}}}$

Thus, if $k \leq \ell < k + n_1$, $v_{|\ell+1|_{n_1}}$ is reachable and if $h \leq \ell < k + n_2 - 1$, $w_{|\ell+1|_{n_2}}$ is also reachable since $({A_1}^{\ell-1} \sqcup\sqcup^1 A_2) B_1 = y_{v_{|\ell+1|_{n_1}}}$ and $({A_1}^{\ell-1} \sqcup\sqcup^1 A_2) B_1 = y_{w_{|\ell+1|_{n_2}}}$, respectively.

Moreover, if $k \leq \ell < n_1$, $v_{|\ell+1|_{n_1}} = v_{\ell+1}$ is reachable and if $n_1 \leq \ell < k + n_1$, $v_{|\ell+1|_{n_1}} = v_{\ell+1-n_1}$ is reached using the unique s-path of composition $(\ell, 1)$, then $I_R(v_\ell) = \ell + n_1$, if $1 \leq \ell \leq k$, and $I_R(v_\ell) = \ell$, if $k+1 \leq \ell < n_1 + 1$. Analogously, $I_R(w_\ell) = \ell$, if $h+1 \leq \ell \leq n_2$ and $I_R(w_\ell) = \ell + n_2$, if $1 \leq \ell \leq k$.

In addition, it is already known that $I_R(w_\ell) = \ell$, if $k < \ell \leq h$. Therefore, all vertices have been reached (before taking two laps on one of the cycles) and hence, the local reachability index of this system is $I_{LR} = k + n_2$.

Example 2: Let (A_1, A_2, B_1, B_2) be the positive 2-D system given in example 1 then 2-D influence digraph is $(2, \{1\}; 3, \{3\})$. Therefore, $c_1 = \max\{k, h\} = h = 3$ and $c_2 = \min\{k + n_1, h + n_2\} = \min\{3, 6\} = 3$ then $c_1 \geq c_2$.

Hence, following the case $c_1 \geq c_2$ one obtains:

$$\begin{aligned} &I_R(v_1) = 3, \quad I_R(v_2) = 2, \\ &I_R(w_1) = 4, \quad I_R(w_2) = 2, \quad I_R(w_3) = 3. \end{aligned}$$

Thus, the local reachability index is $I_{LR} = k + n_2 = 1 + 3 = 4$.

$\boxed{\textit{Second case: } c_1 < c_2}$

Reasoning as in the first case, it is clear that studying all the possible s-paths with the same composition in 2-D influence digraph, one may obtain the reachable vertices and hence, when the system is reachable, the local reachability index of the system.

Similarly to the first case, we start to study the vertices reached by s-paths of composition $(\ell, 0)$ depending on the order relation between $\ell \in \mathbb{N}$ and $c_1' := \min\{k, h\}$, that is:

- If $0 < \ell \leq c_1'$, no vertices are reached by s-paths of composition $(\ell, 0)$ since $({A_1}^{\ell-1} \sqcup\sqcup^0 A_2) B_1 = y_{v_\ell} + y_{w_\ell}$ (see (5)).
- If $c_1' < \ell \leq c_1$, there are two possibilities:
  a) If $c_1' = h < \ell \leq c_1 = k$, the vertex v_ℓ is reached by s-paths of composition $(\ell, 0)$ and $I_R(v_\ell) = \ell$ since (see (6))
  $$({A_1}^{\ell-1} \sqcup\sqcup^0 A_2) B_1 = y_{v_\ell}. \tag{7}$$
  b) If $c_1' = k < \ell \leq c_1 = h$, the vertex w_ℓ is also reached and $I_R(w_\ell) = \ell$ since
  $$({A_1}^{\ell-1} \sqcup\sqcup^0 A_2) B_1 = y_{w_\ell}. \tag{8}$$
- If $c_1 = \max\{k, h\} < \ell$, no vertices are reached by s-paths of composition $(\ell, 0)$ because $({A_1}^{\ell-1} \sqcup\sqcup^0 A_2) B_1 = 0$.

Now, let us study depending on the cycle chosen the vertices reached by s-paths of composition $(\ell - 1, 1)$. Note that each one of these s-paths end either in a vertex $v_{|\ell|_{n_1}}$ with $k < \ell \leq k + n_1$, or in a vertex $w_{|\ell|_{n_2}}$ with $h < \ell \leq h + n_2$. Firstly, let us analyze the s-paths in 1-cycle with such a composition taking $c_2' := c_1' + n_1$, that is:

- If $c_1' = k < \ell \leq c_1$, $v_{|\ell|_{n_1}}$ is reached by s-paths of composition $(\ell - 1, 1)$ and $I_R(v_{|\ell|_{n_1}}) = \ell$ because $({A_1}^{\ell-2} \sqcup\sqcup^1 A_2) B_1 = y_{v_{|\ell|_{n_1}}}$.
- If $c_1 < \ell \leq c_2'$, no vertices are reached by s-paths of composition $(\ell - 1, 1)$ since $({A_1}^{\ell-2} \sqcup\sqcup^1 A_2) B_1 = y_{v_{|\ell|_{n_1}}} + y_{w_{|\ell|_{n_2}}}$.
- If $c_2' < \ell \leq k + n_1$, the vertex $v_{|\ell|_{n_1}}$ is already reached by s-paths of composition $(\ell - n_1, 0)$ (see (7)).

Therefore, to conclude the study of 1-cycle, it is necessary to analyze when the vertex $v_{|\ell|_{n_1}}$ with $c_1 < \ell \leq c_2'$ is reached and hence its local reachability index $I_R(v_{|\ell|_{n_1}})$ may be derived.

Note that for every $v_{|\ell|_{n_1}}$ with $c_1 < \ell \leq c_2'$, there exists an s-path of composition $(\ell - 1, 1)$ ending in it. However, this vertex is not reached by this s-path since there exists another one of the same composition ending in $w_{|\ell|_{n_2}}$. This last statement is due to the fact that an s-path of composition $(\ell - 1, 1)$ (with length ℓ) ends in $w_{|\ell|_{n_2}}$ if and only if $h + 1 \leq \ell \leq h + n_2$ which is always true.

On the whole, the s-paths ending in $v_{|\ell|_{n_1}}$ with $c_1 < \ell \leq c_2'$ have a composition $(\ell - 1 + r(n_1 - 1), 1 + r)$, for each $r \in \mathbb{Z}_+$. In addition, taking into account that the length of such s-paths is $\ell + rn_1$, there exists another s-path with the same composition ending in $w_{|\ell + rn_1|_{n_2}}$ if and only if

$$h + 1 + rn_2 \overset{(a)}{\leq} \ell + rn_1 \overset{(b)}{\leq} h + (r+1)n_2\ . \tag{9}$$

The inequality (9-(b)) is true for all r since $\ell + rn_1 \leq c_2 + rn_1 = \min\{k, h\} + n_1 + rn_1 \leq h + (r+1)n_1 < h + (r+1)n_2$, while (9-$(a)$) is true if and only if $r \leq (\ell - h - 1)/(n_2 - n_1)$.

Thus, there exist no s-paths of composition $(\ell - 1 + r(n_1 - 1), 1 + r)$ ending in a vertex in 2-cycle if $r \in \mathbb{Z}_+$ and $r > (\ell - h - 1)/(n_2 - n_1)$. Then, the vertex $v_{|\ell|_{n_1}}$ is reached using an s-path of composition $(\ell - 1 + r(n_1 - 1), 1 + r)$ with $r = \lfloor (\ell - h - 1)/(n_2 - n_1) \rfloor + 1$. Therefore,

$$I_R(v_{|\ell|_{n_1}}) = \ell + n_1 \left(\left\lfloor \frac{\ell - h - 1}{n_2 - n_1} \right\rfloor + 1 \right)\ . \tag{10}$$

Note that every vertex in 1-cycle has been reached and its associated index calculated. The previous steps are summarized in the following table:

First Cycle	$k+1 \leq \ell \leq k+n_1$	
Case	Composition	$I_R(v_{\vert\ell\vert_{n_1}})$
$k < \ell \leq c_1$	$(\ell-1,1)$	ℓ
$c_1 < \ell \leq c_2'$	$(\ell-1+r(n_1-1),1+r)$ $r=\left\lfloor \frac{\ell-h-1}{n_2-n_1} \right\rfloor+1$	$\ell+rn_1$
$c_2' < \ell \leq k+n_1$	$(\ell-n_1,0)$	$\ell-n_1$

Finally, it is obvious that the highest value of (10) is obtained when $\ell = c_2' = \min\{k,h\}+n_1$. Then, $|\ell|_{n_1} = \min\{k,h\}$ and

$$I_R(v_{\min\{k,h\}}) = \min\{k,h\}+n_1\left\lfloor \frac{\min\{k,h\}+n_2-h-1}{n_2-n_1}+1 \right\rfloor. \quad (11)$$

Secondly, let us study the vertices of the second cycle reached by s-paths of composition $(\ell-1,1)$.

- If $c_1' = h < \ell \leq c_1$, the vertex $w_{|\ell|_{n_2}}$ is reached by s-paths of composition $(\ell-1,1)$ and $I_R(w_{|\ell|_{n_2}}) = \ell$ because $(A_1{}^{\ell-2}\sqcup{}^1A_2)B_1 = y_{w_{|\ell|_{n_2}}}$.
- If $c_1 < \ell \leq c_2$, no vertices are reached by s-paths of composition $(\ell-1,1)$ because $(A_1{}^{\ell-2}\sqcup{}^1A_2)B_1 = y_{v_{|\ell|_{n_1}}} + y_{w_{|\ell|_{n_2}}}$.
- If $c_2 < \ell \leq c_1'+n_2$, $w_{|\ell|_{n_2}}$ is also reached and $I_R(w_{|\ell|_{n_2}}) = \ell$ since $(A_1{}^{\ell-2}\sqcup{}^1A_2)B_1 = y_{w_{|\ell|_{n_2}}}$.
- If $c_1'+n_2 < \ell \leq h+n_2$, the vertex $w_{|\ell|_{n_2}}$ is already reached by s-paths of composition $(\ell-n_2,0)$ (see (8)).

Therefore, to conclude the study of 2-cycle, it is necessary to analyze when the vertex $w_{|\ell|_{n_2}}$ with $c_1 < \ell \leq c_2$ is reached and hence its local reachability index $I_R(w_{|\ell|_{n_2}})$ may be derived.

In general, the s-paths ending in $w_{|\ell|_{n_2}}$ have a composition $(\ell-1+r(n_1-1),1+r)$, for every $r \in \mathbb{Z}_+$. Moreover, taking into consideration that the length of these s-paths is $\ell+rn_2$, there exists another s-path with the same composition ending in $v_{|\ell+rn_2|_{n_1}}$ if and only if

$$k+1+rn_1 \overset{(a)}{\leq} \ell+rn_2 \overset{(b)}{\leq} k+(r+1)n_1 \ . \quad (12)$$

The inequality (12-(a)) is true for all r since $\ell > k$ and $n_1 < n_2$, while (12-(b)) is true if and only if $r \leq (k+n_1-\ell)/(n_2-n_1)$.

Thus, there exist no s-paths of composition $(\ell-1+r(n_2-1),1+r)$ ending in a vertex in 1-cycle if $r \in \mathbb{Z}_+$ and $r > (k+n_1-\ell)/(n_2-n_1)$. Then, the vertex $w_{|\ell|_{n_2}}$ is reached using an s-path of composition $(\ell-1+r(n_1-1),1+r)$ with $r = \lfloor (k+n_1-\ell)/(n_2-n_1) \rfloor + 1$. Therefore,

$$I_R(w_{|\ell|_{n_2}}) = \ell+n_2\left(\left\lfloor \frac{k+n_1-\ell}{n_2-n_1} \right\rfloor+1\right) \ . \quad (13)$$

Note that every vertex in 2-cycle has been reached and its associated index calculated which is summarized as follows:

Second Cycle	$h+1 \leq \ell \leq h+n_2$	
Case	Composition	$I_R(w_{\vert\ell\vert_{n_2}})$
$h < \ell \leq c_1$	$(\ell-1,1)$	ℓ
$c_1 < \ell \leq c_2$	$(\ell-1+r(n_2-1),1+r)$ $r=\left\lfloor \frac{k+n_1-\ell}{n_2-n_1} \right\rfloor+1$	$\ell+rn_2$
$c_2 < \ell \leq c_1'+n_2$	$(\ell-1,1)$	ℓ
$c_1'+n_2 < \ell \leq h+n_2$	$(\ell-n_2,0)$	$\ell-n_2$

Following, it is analyzed which is the vertex $w_{|\ell|_{n_2}}$ with $c_1 < \ell \leq c_2$ to provide the maximum local reachability index of the vertices in 2-cycle and for that, it is shown which is the maximum $\widetilde{\ell} \in \mathbb{N}$ with $c_1 < \widetilde{\ell} \leq c_2$ leading to the maximum length of s-paths reaching $w_{|\widetilde{\ell}|_{n_2}}$.

To simplify, let us consider $r_\ell := \left\lfloor \frac{n_1+k-\ell}{n_2-n_1} \right\rfloor$. Obviously, the highest value $\widetilde{r}$ of r_ℓ is obtained when $\ell = c_1+1$ as $c_1 < \ell \leq c_2$ i.e. $\widetilde{r} = r_{c_1+1}$.

Let us define

$$\widetilde{\ell} := n_1+k-(n_2-n_1)\widetilde{r}. \quad (14)$$

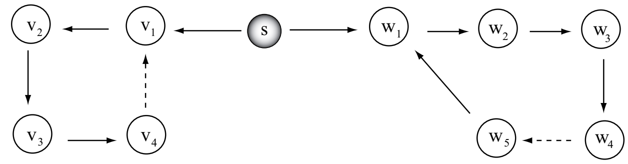

Fig. 3. Digraph for system in example 3.

Thus, $\widetilde{\ell} \leq n_1+k$ and $c_1 < \widetilde{\ell} \leq c_2$ because $k \leq h+n_1-1$ and

$$\begin{array}{rcl} \widetilde{\ell} & \leq & n_1+h+(n_1-1)-(n_2-n_1)\lfloor (n_1-1)/(n_2-n_1) \rfloor \\ & = & n_1+h+((n_1-1) \bmod (n_2-n_1)) \leq h+n_2-1. \end{array}$$

if $n_2+h < n_1+k$. In addition, $r_{\widetilde{\ell}} = \left\lfloor \frac{n_1+k-\widetilde{\ell}}{n_2-n_1} \right\rfloor = \frac{n_1+k-\widetilde{\ell}}{n_2-n_1} = \widetilde{r}$,

$$I_R(w_{|\widetilde{\ell}|_{n_2}}) = k+n_1\left\lfloor \frac{k+n_2-\max\{k,h\}-1}{n_2-n_1} \right\rfloor+n_2, \quad (15)$$

and $I_R(w_{|\widetilde{\ell}|_{n_2}}) \geq I_R(w_{|\ell|_{n_2}})$ for all ℓ, $c_1 < \ell \leq \widetilde{\ell}$.

Lastly, when $\ell > \widetilde{\ell}$, $\ell = \widetilde{\ell}+t(n_2-n_1)+a$ taking $t := \left\lfloor \frac{\ell-\widetilde{\ell}}{n_2-n_1} \right\rfloor$ and $a := (\ell-\widetilde{\ell}) \bmod (n_2-n_1)$. Furthermore,

$$r_\ell = \left\lfloor \widetilde{r}-t-\frac{a}{n_2-n_1} \right\rfloor = \begin{cases} \widetilde{r}-t-1, & \text{if } a \neq 0; \\ \widetilde{r}-t, & \text{if } a = 0, \end{cases}$$

then $r_\ell < \widetilde{r}$ since t and a cannot be simultaneously zero. When so, it is obvious that $I_R(w_{|\ell|_{n_2}}) < I_R(w_{|\widetilde{\ell}|_{n_2}})$. Hence, $I_R(w_{|\ell|_{n_2}}) \leq I_R(w_{|\widetilde{\ell}|_{n_2}})$ for all ℓ such that $c_1 < \ell \leq c_2$. Besides that, $I_R(w_{|\widetilde{\ell}|_{n_2}})$ given in (15) is strictly greater than $I_R(v_{\min\{k,h\}})$ calculated in (11) as $k-\max\{k,h\} = \min\{k,h\}-h$. Therefore, for this special class of systems,

$$I_{LR} = k+n_1\left\lfloor \frac{k+n_2-\max\{k,h\}-1}{n_2-n_1} \right\rfloor+n_2. \quad (16)$$

Example 3: Let (A_1,A_2,B_1,B_2) be a positive 2-D system given by

$$A_1 = \left[\ e_2 \mid e_3 \mid e_4 \mid 0 \mid e_6 \mid e_7 \mid e_8 \mid 0 \mid e_5\ \right] \in \mathbb{R}_+^{9\times 9},$$

$$A_2 = \left[\ 0 \mid 0 \mid 0 \mid e_1 \mid 0 \mid 0 \mid 0 \mid e_9 \mid 0\ \right] \in \mathbb{R}_+^{9\times 9},$$

$B_1 = \left[\ e_1+e_5\ \right] \in \mathbb{R}_+^{9\times 1}$ and $B_2 = 0 \in \mathbb{R}_+^{9\times 1}$, with 2-D influence digraph $(4,\{4\};5,\{4\})$ corresponding to Fig. 3.

Therefore, $c_1 = \max\{k,h\} = 4$ and $c_2 = \min\{k+n_1,h+n_2\} = 9$. Hence, following the case $c_1 < c_2$ it is obtained

$$I_R(v_1) = 9,\ I_R(v_2) = 14,\ I_R(v_3) = 19,\ I_R(v_4) = 24,$$

$$I_R(w_1) = 21,\ I_R(w_2) = 17,\ I_R(w_3) = 13,\ I_R(w_4) = 9,\ I_R(w_5) = 25.$$

Thus, the local reachability index is

$$\begin{array}{rcl} I_{LR} & = & k+n_1\left\lfloor \dfrac{k+n_2-\max\{k,h\}-1}{n_2-n_1} \right\rfloor+n_2 \\ & = & 4+4\left\lfloor \dfrac{4+5-4-1}{5-4} \right\rfloor+5 = 25 = I_R(w_5). \end{array}$$

Finally, let us verify that the above formula obtained in the case $c_1 < c_2$ is also useful for the first case $c_1 \geq c_2$.

As we have seen, when $c_1 \geq c_2$ then $h = \max\{k,h\} \geq k+n_1$. Consequently,

$$\begin{array}{l} k+n_2-\max\{k,h\}-1 = k+n_2-h-1 \\ \quad = (k+n_1-h)+(n_2-n_1-1) \leq n_2-n_1-1. \end{array}$$

Hence, $\left\lfloor \frac{k+n_2-\max\{k,h\}-1}{n_2-n_1} \right\rfloor = 0$, and when so, the expression (16) is reduced to $I_{LR} = k+n_2$, which is the desired aim. Summing-up, the local reachability index of the system with 2-D influence digraph $(n_1,\{k\};n_2,\{h\})$ is given in all cases by expression (16).

Afterwards, the following result holds.

Theorem 1: Let (A_1,A_2,B_1,B_2) be a positive 2-D system with 2-D influence digraph $(n_1,\{k\};n_2,\{h\})$. Then, this system is reachable if and only if $n_1 \neq n_2$ and so the local reachability index I_{LR} is given by

$$I_{LR} = k + \tilde{n}_1 \left\lfloor \frac{k+\tilde{n}_2 - \max\{k,h\} - 1}{\tilde{n}_2 - \tilde{n}_1} \right\rfloor + \tilde{n}_2,$$

where $\tilde{n}_1 = \min\{n_1,n_2\}$ and $\tilde{n}_2 = \max\{n_1,n_2\}$.

Proof: If $n_1 = n_2$, Lemma 1 establishes that the system is not reachable. For the case $n_1 < n_2$, the local reachability index of the system $(n_1,\{k\};n_2,\{h\})$ is given by expression (16) as it has been analyzed formerly. Finally, the case $n_1 > n_2$ is reduced to the previous case taking the system (A_2,A_1,B_1,B_2) with 2-D influence digraph $(n_2,\{h\};n_1,\{k\})$. ■

Note that for this class of special systems the local reachability index always appears to be associated with the vertices of the second cycle $w_{|\tilde{j}|_{n_2}}$, being $\tilde{j}$ given by expression (14). Besides that, if $h \leq k$, the value of h does not influence the local reachability index.

Lemma 2: Let (A_1,A_2,B_1,B_2) be a positive 2-D system with 2-D influence digraph $(n_1,\{k\};n_2,\{h\})$, being n_1 and n_2 fixed natural numbers with $n_1 < n_2$ and $n_1 + n_2 = n$. Then, the maximum local reachability index is achieved by those systems satisfying $k = n_1$ and $h \leq k$. Moreover, under these circumstances,

$$I_{LR} = n + n_1 \left\lfloor \frac{n_2 - 1}{n_2 - n_1} \right\rfloor. \tag{17}$$

Proof: Since $k + n_2 - \max\{k,h\} - 1$ is greatest when $\max\{k,h\} = k$ and when so, $k + n_2 - \max\{k,h\} - 1 = n_2 - 1$. Then by $k \leq n_1$ and $n_1 + n_2 = n$, the results are held. ■

Theorem 2: Any systems (A_1,A_2,B_1,B_2) with 2-D influence digraph $(n_1,\{k\};n_2,\{h\})$ where $n_1 \neq n_2$ and $= n_1 + n_2$ satisfy the condition that its local reachability index I_{LR} is upper bounded by $\frac{(n+1)^2}{4}$. In addition, for every odd natural number n there exists at least one system with 2-D influence digraph $(n_1,\{k\};n_2,\{h\})$ such that its local reachability index is equal to this upper bound.

Proof: The maximum value of (17) occurs if $n_2 - n_1 = 1$. In this case $n = n_1 + n_2 = 2n_1 + 1$ is an odd natural number and $n_1 = \frac{(n-1)}{2}$, $n_2 = \frac{(n+1)}{2}$. Then,

$$I_{LR} = n + n_1 \left\lfloor \frac{n_2 - 1}{n_2 - n_1} \right\rfloor = n + n_1^2 = n + \frac{(n-1)^2}{4} = \frac{(n+1)^2}{4}.$$

If n is an even natural number, the minimum value of $n_2 - n_1$ is 2 and it occurs when $n_1 = \frac{n}{2} - 1$ and $n_2 = \frac{n}{2} + 1$. In this case,

$$I_{LR} = n + n_1 \left\lfloor \frac{n_2 - 1}{n_2 - n_1} \right\rfloor = n + \left(\frac{n}{2} - 1\right) \left\lfloor \frac{n}{4} \right\rfloor. \tag{18}$$

Notice that $n = n_1 + n_2 \geq 3$ as $n_1 \neq n_2$ and if n is an even natural number then $n \geq 4$. Hence, if n is a multiple of 4 i.e. $n = 4p$, $p \in \mathbb{N}$, then $\left\lfloor \frac{n}{4} \right\rfloor = p = \frac{n}{4}$ and so (18) is equal to $\frac{n^2+6n}{8}$. Finally, if n is even but not multiple of 4 that is $n = 4p+2$, $p \in \mathbb{N}$, then $\left\lfloor \frac{n}{4} \right\rfloor = p = \frac{n-2}{4}$ and so (18) is equal to $\frac{(n+2)^2}{8}$. To shorten,

$$\max\{I_{LR}\} = \begin{cases} \dfrac{(n+1)^2}{4}, & \text{if } n \text{ is an odd number,} \\ \dfrac{n^2+6n}{8}, & \text{if } n \text{ is a multiple of 4 and} \\ \dfrac{(n+2)^2}{8}, & \text{otherwise.} \end{cases}$$

It is obvious that $\frac{(n+2)^2}{8} \leq \frac{(n^2+6n)}{8} < \frac{(n+1)^2}{4}$, $\forall n \geq 2$. Hence, $I_{LR} \leq \frac{(n+1)^2}{4}$, which completes the proof. ■

IV. Remarks on the local reachability index

Let us see that the upper bound obtained in this paper is only valid for the class of the chosen systems. There are many examples whose local reachability index is greater than $\frac{(n+1)^2}{4}$. In the same way, such examples show us that all well-known conjectures are not valid for any positive 2-D systems.

Example 4: Let the system

$$(A_1,A_2,B_1,B_2) = \left(\begin{bmatrix} 0 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \right)$$

with 2-D influence digraph corresponding to Fig. 4, note that $I_R(v_1) =$

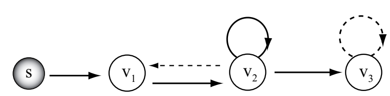

Fig. 4. Digraph for system in example 4.

1, $I_R(v_2) = 2$ and $I_R(v_3) = 6$ since all s-paths of composition $(1,0)$, $(2,0)$ and $(3,3)$ end in the single vertex v_1, v_2 and v_3, respectively. Thus, the system is locally reachable and $I_{LR} = 6$. Therefore, the upper bound $\frac{(n+1)^2}{4}$ given in the precedent section is not useful for this system since $I_{LR} = 6 > 4 = \frac{(n+1)^2}{4}$ if $n = 3$.

Therefore, an upper bound of the local reachability index of any positive 2-D system must be greater than $\frac{(n+1)^2}{4}$, which shows that the conjecture given in [1] fails.

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