On the Reachability Index of Positive 2-D Systems

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Abstract— The structural property of local reachability for positive 2-D systems refers to single local states. The smallest number of steps needed to reach all local states of a system is the local reachability index of the system. This index may exceed the system dimension. Some authors have studied upper bounds on the local reachability index for specific positive 2-D systems and have suggested different upper bounds for any positive 2-D system. In this paper, the local reachability index for a special class of positive 2-D systems is characterized and an upper bound for this index is derived. A comparison with previous results is presented.

Index Terms— Positive two dimensional (2-D) systems, reachability, Hurwitz products, influence digraph, local reachability index, nonnegative matrices.

I. INTRODUCTION

During the last decade, the theory of the positive two-dimensional systems has been considerably enhanced by the study of different physical problems (see [1], [2] and [3]).

One of the most frequent representations of positive 2-D systems is the Fornasini-Marchesini model (see [4] and [1]) which is as follows:

\[ x_{i+j+1} = A_1 x_{i+j} + A_2 x_{i+j-1} + B_1 u_{i+j} + B_2 u_{i+j-1} \]

where local states \( x(i,j) \in \mathbb{R}_{++}^n \), inputs \( u(i,j) \in \mathbb{R}_{++}^m \), \( A_1, A_2 \in \mathbb{R}_{++}^{n \times n} \), \( B_1, B_2 \in \mathbb{R}_{++}^{n \times m} \) and initial global state \( x_0 := \{x(h,k) : (h,k) \in \mathbb{G}_0\} \) being \( \mathbb{G}_0 := \{(h,k) : h,k \in \mathbb{Z}, h+k = 0\} \) the separation set. Let us denote this system by \((A_1, A_2, B_1, B_2)\).

The smallest number of steps needed to reach all local states of the system is the local reachability index of that system. The reachability index in positive 1-D systems, which is always bounded by \( n \) (see [5], [6]), has been studied in [7], [8], [9], [10] and [11]. However, on characterizing the reachability index seems to be a hard task for a positive 2-D system.

In [12], the authors suggested \( \frac{n^2}{2} \) as an upper bound for the local reachability index of every positive 2-D system. Later on, in [1], the same authors reviewed the aforementioned conjecture suggesting \( \frac{(n+1)^2}{2} \) as a new upper bound.

Before [1] was published, Kaczorek (see [13]) reviewed [12] and checked that the upper bound \( \frac{n^2}{2} \) fails with an example. Moreover, in that same paper, the author stated that \( 2(n+1) \) is an upper bound for the local reachability index of the \( n \)th order positive 2-D general models. Hence, such an upper bound is also useful for Fornasini-Marchesini Models (see [4]) since these kinds of systems are a particular case of \( n \)th order positive 2-D general models. However, since \( \frac{(n+1)^2}{2} \) is greater than \( 2(n+1) \) for all \( n \geq 8 \), if the first bound fails, the second one necessarily does.

The paper has been organized as follows: Section II introduces some notations and basic definitions used in the paper. In Section III, the local reachability index for a special class of positive 2-D systems is completely characterized. Moreover, for this class of systems is checked that the corresponding indices are always bounded by \( \frac{(n+1)^2}{4} \), they even turn to be \( \frac{(n+1)^2}{2} \) in suitable conditions.

II. NOTATIONS AND PRELIMINARY DEFINITIONS

Denote by \( |z| = 1 + (i - j) \mod n \), \( i \in \mathbb{N} \); by \( |z| \) the lower integral-part of \( z \in \mathbb{R} \) and by \( \text{col}_j(A) \) the \( j \)th column of the matrix \( A \).

Definition 1: Hurwitz products of the \( n \times n \) matrices, \( A_1 \) and \( A_2 \) are defined as

\[
A_1^{(i)} A_2 = 0, \quad \text{when either } i \text{ or } j \text{ is negative},
\]

\[
A_1^{(i)} A_2 = A_1, \quad \text{if } i \geq 0,
\]

\[
A_1^{(i)} A_2 = A_2, \quad \text{if } j \geq 0.
\]

Note that \( \sum_{i,j=1}^{n} A_1^{(i)} A_2 = (A_1 + A_2) \).

Definition 2: (see [1]) A 2-D state-space model (1) is (positively) locally reachable if, upon assuming \( z_0 = 0 \), for every \( x^* \in \mathbb{R}^n \), there exists \((h,k) \in \mathbb{Z} \times \mathbb{Z}, h+k > 0\) and a nonnegative input sequence \( u(\cdot, \cdot) \) such that \( x(h,k) = x^* \). When the state is said to be (positively) reachable in \( h+k \) steps. The smallest number of steps that allows to reach every nonnegative local state represents the local reachability index \( LR \) of such system.

Local reachability is equivalent to the possibility of reaching every vector of the standard basis of \( \mathbb{R}^n \) or equivalently to a corresponding positive monomial vector, that is, any positive multiples of the \( n \)th vector of the standard basis of \( \mathbb{R}^n \). Denote by \( y_i \) any \( i \)th positive monomial vectors. In the same way, a monomial matrix is a nonsingular matrix having a unique positive entry in each row and column.

Therefore, the study of local reachability can be reduced to the analysis of the reachability matrix in \( k \)-steps (see [1])

\[
\mathcal{R}_k = [\mathcal{R}_{k-1}] A_1 A_2 + [\mathcal{R}_{k-1}] A_2 B_1 A_1 B_2 + [\mathcal{R}_{k-1}] A_2 B_1 A_1 B_2 A_2 B_3 + \cdots + A_2^{k-1} B_2]
\]

when \( k \) varies over \( \mathbb{N} \) since the local reachability property is held if and only if there exist \( n \) pairs \((h_i,k_i), h_i,k_i \in \mathbb{N} \times \mathbb{N}, i = 1, \ldots, n\), and \( n \) indices \( j \neq j(i) \in \{1, 2, \ldots, m\} \) such that \((A_1^{j-1} A_2 B_1 + A_2^{j-1} A_1 B_2) \mathcal{R}_{j(i)-1} \) is an \( n \)th monomial vector, that is, there exists \( k \in \mathbb{N} \) such that \( \mathcal{R}_k \) contains an \( n \times n \) monomial matrix.

Definition 3: (see [1]) Associated with system (1), a directed digraph called 2-D influence digraph is defined. It is denoted by \( \mathcal{D}(A_1, A_2, B_1, B_2) \), which is given by \( \mathcal{S}(V, \mathcal{A}_1, \mathcal{A}_2, \mathcal{B}_1, \mathcal{B}_2) \), where \( V = \{v_1, v_2, \ldots, v_n\} \) is the set of vertices, \( \mathcal{A}_1 \) and \( \mathcal{A}_2 \) are subsets of \( \mathcal{V} \times \mathcal{V} \times \mathcal{V} \times \mathcal{V} \) whose elements are called \( \mathcal{A}_1 \)-arcs and \( \mathcal{A}_2 \)-arcs respectively, while \( \mathcal{B}_1 \) and \( \mathcal{B}_2 \) are subsets of \( \mathcal{V} \times \mathbf{V} \) whose elements are called \( \mathcal{B}_1 \)-arcs and \( \mathcal{B}_2 \)-arcs respectively. There is an \( \mathcal{A}_1 \)-arc (\( \mathcal{A}_2 \)-arc) from \( v_i \) to \( v_j \) if and only if (the \( i,j \)th entry of \( A_1 \) (\( A_2 \)) is nonzero. There is a \( \mathcal{B}_1 \)-arc (\( \mathcal{B}_2 \)-arc) from \( s_j \) to \( v_j \) if and only if (the \( j \)th entry of \( B_1 \) (\( B_2 \)) is nonzero.

A path in \( \mathcal{D}(A_1, A_2, B_1, B_2) \) from \( v_i \) to \( v_j \) is a sequence of adjacent arcs (i.e. \( (v_i, v_{i_1}), (v_{i_1}, v_{i_2}), \ldots, (v_{i_j}, v_j) \)). In particular, an \( s_j \)-path is a path originating from the source \( s_j \).

Denote by \( p(q) \) the number of 1-arcs (2-arcs) occurring in a path \( \mathcal{P} \). The pair \( (p,q) \) is called the composition of \( \mathcal{P} \) and \( p+q \) its length.

A circuit is defined to be a path whose extreme vertices coincide and if each vertex appears exactly once as the first vertex of an arc, the circuit is said to be a cycle.

A vertex \( v_i \) is called reachable in \( p+q \) steps (briefly reachable) if the corresponding \( n \)th monomial vector is reachable in \( p+q \) steps (\( y_i \) appears in the reachability matrix in \( (p+q) \)-steps). That is (see [1]), from the combinatorial point of view, there exist \( q_p \in \mathbb{Z}^+ \), \( 0 < p < q \), such that all \( s \)-paths of composition \( (p,q) \) finish in the same vertex \( v_i \) (\( (v_{i_1}, v_{i_2}), \ldots, (v_{i_j}, v_j) \)) and from the matrix point of view, \( (A_1^{p+q} A_2^{q-p} B_1 + A_2^{p+q} A_1^{q-p} B_2) y_i = y_i \), for some \( j \in \{1, \ldots, m\} \). Denote by \( \mathcal{L}_p(v_i) \) the minimum length of the \( s \)-paths reaching \( v_i \).
III. LOCAL REACHABILITY INDEX FOR A SPECIAL CLASS OF SYSTEMS

Let us consider an nth order positive 2-D system \((A_1, A_2, B_1, B_2)\) with \(B_2 = 0\). \(B_1\) a column vector with just two positive entries and 2-D influence digraph of this system consists of two loops corresponding to that unique source \(s\) with \(n_1\) \((v_1, \ldots, v_{n_1})\) and \(n_2\) \((w_1, \ldots, w_{n_2})\) vertices respectively and with all 1-arcs except for 2-arcs \((v_k, v_{k+1})\) and \((w_h, w_{h+1})\). That is, \((A_1, A_2, B_1, B_2)\) is similar under permutation matrices to \((\hat{A}_1, \hat{A}_2, \hat{B}_1, \hat{B}_2)\) being
\[
\hat{A}_1 = \begin{bmatrix} \hat{A}_1' & 0 \\ 0 & \hat{A}_1'' \end{bmatrix}, \quad \hat{A}_2 = \begin{bmatrix} \hat{A}_2' & 0 \\ 0 & \hat{A}_2'' \end{bmatrix}, \quad \hat{B}_1 = [y_1 + y_{n_1+1}] \quad \text{and} \quad \hat{B}_2 = 0,
\]
where both \(\hat{A}_1' \in \mathbb{R}^{n_1 \times n_1}\) and \(\hat{A}_1'' \in \mathbb{R}^{n_1 \times n_1}\) have the following structure
\[
\begin{pmatrix}
0 & \cdots & 0 & \cdots & 0 \\
\vdots & \ddots & \vdots & \ddots & \vdots \\
0 & \cdots & 0 & \cdots & 0 \\
0 & \cdots & 0 & \cdots & 0 \\
0 & \cdots & 0 & \cdots & 0
\end{pmatrix}
\]
except for the case when both \(\hat{A}_1''\) and \(\hat{A}_1''\) have the structure of the upper left block of (2) and both \(\hat{A}_2' \in \mathbb{R}^{n_2 \times n_2}\) and \(\hat{A}_2'' \in \mathbb{R}^{n_2 \times n_2}\) have the following structure
\[
\begin{pmatrix}
0 & \cdots & 0 & \cdots & 0 \\
\vdots & \ddots & \vdots & \ddots & \vdots \\
0 & \cdots & 0 & \cdots & 0 \\
0 & \cdots & 0 & \cdots & 0 \\
0 & \cdots & 0 & \cdots & 0
\end{pmatrix}
\]
except for the case when both \(\hat{A}_2'\) and \(\hat{A}_2''\) have the structure of the lower left block of (3), \(+\) denoting a strictly positive entry. Moreover, 2-D influence digraph is illustrated in Fig. 1 where continuous (dotted) arrows represent 1-arcs (2-arcs).

For this class of systems we concisely denote 2-influence digraph by means of the quadruple \((n_1, \{k\}; n_2, \{h\})\) with \(k (h)\) indicating where the single 2-arc of the first (second) cycle is located. In addition, \(n_1\) and \(n_2\) indicating the number of vertices of each cycle respectively and \(n = n_1 + n_2\). Finally, the vertices \(v_{n_1+1}, v_{n_1+2}, \ldots, v_{n_1+n_2}\) are relabelled as \(w_1, w_2, \ldots, w_{n_2}\), to distinguish the vertices of both cycles.

Example 1: The positive 2-D system described by the matrices
\[
(A_1, A_2, B_1, B_2) = \begin{bmatrix}
1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 \\
0 & 0 & 1 & 0
\end{bmatrix}
\]
has a 2-D influence digraph \((2, \{1\}; 3, \{3\})\), with \(n_1 = 2, n_2 = 3, k = 1\) and \(h = 3\), given in Fig. 2.

Firstly, let us give a necessary condition for local reachability.

Lemma 1: Let a locally reachable system with 2-D influence digraph \((n_1, \{k\}; n_2, \{h\})\) then the lengths of the cycles are different; i.e., \(n_1 \neq n_2\).

Proof: Note that every \(s\)-path \(P\) finishing in the vertex \(v_1\) has a composition \(1 + r(n_1 - 1), r\) for some \(r \in \mathbb{Z}_+\). Furthermore, if \(n_1 = n_2\), it is clear that there exists another \(s\)-path with the same composition of \(P\) finishing in the vertex \(w_1\). Hence, the vertex \(v_1\) is not reachable.

Secondly, let us verify, in a constructive way, that a system with 2-D influence digraph \((n_1, \{k\}; n_2, \{h\})\) having \(n_1 \neq n_2\) is locally reachable. At the same time, we will obtain the local reachability index upon assuming without a loss of generality that \(n_1 < n_2\). Furthermore, this index \(I_{\text{LR}}\) will be deduced in accordance to the order of \(c_1\) and \(c_2\) being \(c_1 := \max\{k, h\}\) and \(c_2 := \min\{k + n_1, h + n_2\}\).

Note that
\[
k < k + n_1, \quad k \leq n_1 < n_2 < h + n_2 \quad \text{and} \quad h < h + n_2.
\]

First case: \(c_1 = c_2\)

In this case, considering both \(c_1 \geq c_2\) and (4), one finds that \(c_1 = h\), \(c_2 = k + n_1\) and \(k < k + n_1 \leq h < h + n_2\).

Let us study the reachable vertices \(v_i\) and their corresponding indices \(I_{\text{LR}}(v_i)\). The reachable vertices will be deduced studying all the possible \(s\)-paths with the same composition in 2-D influence digraph, that is, calculating step by step the different products of Hurwitz’s appearing in the reachability matrices, \((A_1^{(-1)}, A_2^{(-1)})B_1\), \((A_1^{(-1)}, A_2^{(-1)})B_2\), and so on.

To facilitate the reading, let us use a schematic representation. On the left side, let us indicate the \(s\)-paths of a given composition, more specifically let us point out the cycle chosen to construct the \(s\)-path and those arcs involved. On the other side, let us indicate the Hurwitz products pertaining to the \(s\)-path on the left side. To simplify, we will denote by \(v_i\) and \(y_i\) the monomial vectors \(v_i\) and \(y_i\) respectively.

If \(0 < \ell \leq k\), no vertices are reached by \(s\)-paths of composition \((\ell, 0)\) (see Fig. 1) since

\[
\begin{array}{l}
\text{s-path of Composition } (\ell, 0) \\
\text{Corresponding Hurwitz Products}
\end{array}
\]

In 1-cycle
\[
\begin{array}{l}
(s, v_1), (v_1, v_2), \ldots, (v_{\ell-1}, v_\ell) \\
(A_1^{(-1)}, A_2^{(-1)})y_{v_1} = (A_1^{(-1)}, A_2^{(-1)})y_{v_1} = y_{v_1}
\end{array}
\]

In 2-cycle
\[
\begin{array}{l}
(s, w_1), (w_1, w_2), \ldots, (w_{\ell-1}, w_\ell) \\
(A_1^{(-1)}, A_2^{(-1)})y_{w_1} = (A_1^{(-1)}, A_2^{(-1)})y_{w_1} = y_{w_1}
\end{array}
\]

then
\[
(A_1^{(-1)}, A_2^{(-1)})B_1 = (A_1^{(-1)}, A_2^{(-1)})y_{v_1} + y_{w_1} = y_{v_1} + y_{w_1}.
\]

However, if \(k < \ell \leq h\), the vertex \(w_{\ell}\) is reachable since

\[
\begin{array}{l}
\text{s-path of Composition } (\ell, 0) \\
\text{Hurwitz Products}
\end{array}
\]

In 1-cycle
\[
\begin{array}{l}
(s, w_1), (w_1, w_2), \ldots, (w_{\ell-1}, w_\ell) \\
(A_1^{(-1)}, A_2^{(-1)})y_{w_1} = (A_1^{(-1)}, A_2^{(-1)})y_{w_1} = y_{w_1}
\end{array}
\]

In 2-cycle
\[
\begin{array}{l}
(s, w_1), (w_1, w_2), \ldots, (w_{\ell-1}, w_\ell) \\
(A_1^{(-1)}, A_2^{(-1)})y_{w_1} = (A_1^{(-1)}, A_2^{(-1)})y_{w_1} = y_{w_1}
\end{array}
\]

then
\[
(A_1^{(-1)}, A_2^{(-1)})B_1 = (A_1^{(-1)}, A_2^{(-1)})y_{v_1} + y_{w_1} = y_{w_1}.
\]
Note that $I_R(w_\ell) = \ell$ if $k < \ell \leq h$. Moreover, no other vertices are reached by $s$-paths of composition $(\ell, 0)$ if $h < \ell$ since $(A_1^{(\ell-1), 1, 0}A_2)B_1 = 0$.

Likewise, using $c_2 = k + n_1 \leq h = c_1$ and knowing that if $i = \ell$, $\forall \ell \geq 0$, Hurwitz products are:

$$(A_1^{(\ell-1), 1, 1}A_2)B_1 = (A_1^{(\ell-1), 1, 0}A_1^{2, -1}A_2A_1 + \ldots + A_1A_2A_1^{2, -2} + A_2A_1^{2, -1})B_1;$$

then,

if $0 \leq \ell \leq k - 1$ :

$s$-path of Composition $(\ell, 1)$

<table>
<thead>
<tr>
<th>Hurwitz Products</th>
</tr>
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<tbody>
<tr>
<td>$A_1^{(\ell-1), 1, 1}A_2$ when $y_{i+1} = 0$</td>
</tr>
</tbody>
</table>

In 1-cycle

No paths (Need k 1-arcs)

In 2-cycle

No paths (Need h 1-arcs)

if $k \leq \ell < k + n_1$ :

$s$-path of Composition $(\ell, 1)$

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</table>

In 1-cycle

$(s, v_1), (s, v_2), \ldots, (s, v_{i+1}), \ldots, (s, v_{i+1}) (A_1^{(\ell-1), 1, 1}A_2)v_{i+1} = y_{i+1}$

In 2-cycle

No paths (Need h 1-arcs)

if $k + n_1 \leq \ell < h$ :

$s$-path of Composition $(\ell, 1)$

<table>
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</thead>
<tbody>
<tr>
<td>$A_1^{(\ell-1), 1, 1}A_2$ when $y_{i+1} = 0$</td>
</tr>
</tbody>
</table>

In 1-cycle

No paths (Need two 2-arcs)

In 2-cycle

No paths (Need h 1-arcs)

if $h \leq \ell < k + n_1 - 1$ :

$s$-path of Composition $(\ell, 1)$

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<tr>
<td>$A_1^{(\ell-1), 1, 1}A_2$ when $y_{i+1} = 0$</td>
</tr>
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</table>

Thus, if $k \leq \ell < k + n_1$, $v_{i+1}$ is reachable and if $h \leq \ell < k + n_2 - 1$, $w_{i+1}$ is also reachable since $(A_1^{(\ell-1), 1, 1}A_2)B_1 = y_{i+1}$ and $(A_1^{(\ell-1), 1, 1}A_2)B_1 = y_{i+1}$, respectively.

Moreover, if $k \leq \ell < n_1$, $v_{i+1}$ is reachable and if $n_1 \leq \ell < k + n_1$, $v_{i+1}$ is also reachable since $I_R(w_\ell) = \ell$, $I_R(w_\ell) = \ell$, and $I_R(v_\ell) = \ell$, if $h \leq \ell < n_1 + 1$. Analogously, $I_R(w_\ell) = \ell$, $h + 1 \leq \ell \leq n_2$ and $I_R(w_\ell) = \ell$, if $1 \leq \ell < k$.

In addition, it is already known that $I_R(w_\ell) = \ell$, if $k < \ell \leq h$.

Therefore, all vertices have been reached (before taking two loops on one of the cycles) and hence, the local reachability index of this system is $I_R = k + n_2$.

Example 2: Let $(A_1, A_2, B_1, B_2)$ be the positive 2-D system given in example 1 then 2-D influence digraph is $(2, \{1, 3, 3\})$. Therefore, $c_1 = \min \{k, h\} = h = 3$ and $c_2 = \min \{k + n_1, h + n_2\} = \min \{3, 6\} = 3$ then $c_1 \geq c_2$.

Hence, following the case $c_1 \geq c_2$ one obtains:

$I_R(v_1) = 3, \quad I_R(v_2) = 2, \quad I_R(w_1) = 4, \quad I_R(w_2) = 2, \quad I_R(w_3) = 3$.

Thus, the local reachability index is $I_R = k + n_2 = 1 + 3 = 4$.

Second case: $c_1 < c_2$

Reasoning as in the first case, it is clear that studying all the possible $s$-paths with the same composition in 2-D influence digraph, one may obtain the reachable vertices and hence, when the system is reachable, the local reachability index of the system.

Similarly to the first case, we start to study the vertices reached by $s$-paths of composition $(\ell, 0)$ depending on the order relation between $\ell \in \mathbb{N}$ and $c_1 := \min \{k, h\}$, that is:

- If $0 < \ell < c_1$, no vertices are reached by $s$-paths of composition $(\ell, 0)$ since $(A_1^{(\ell-1), 1, 0}A_2)B_1 = y_{i+1}$ (see (5)).
- If $c_1 < \ell \leq c_1$, there are two possibilities:
  - a) If $c_1 < \ell < c_1 - k$, the vertex $v_1$ is reached by $s$-paths of composition $(\ell, 0)$ and $I_R(v_1) = \ell$ (since (6))
    $$(A_1^{(\ell-1), 1, 0}A_2)B_1 = y_{i+1}. \quad (7)$$
  - b) If $c_1 < \ell < c_1 - k + n_1$, the vertex $v_2$ is also reached and
    $I_R(w_2) = \ell$ since
    $$(A_1^{(\ell-1), 1, 0}A_2)B_1 = y_{i+1}. \quad (8)$$
- If $c_1 = \max \{k, h\} < \ell$, no vertices are reached by $s$-paths of composition $(\ell, 0)$ because $(A_1^{(\ell-1), 1, 0}A_2)B_1 = 0$.

Now, let us study depending on the cycle chosen the vertices reached by $s$-paths of composition $(\ell - 1, 1)$. Note that each one of these $s$-paths end either in a vertex $v_{i+1}$, with $k < \ell < k + n_1$, or in a vertex $w_{i+1}$, with $h < \ell < h + n_2$. Firstly, let us analyze the $s$-paths in 1-cycle with such a composition taking $c_2 := c_1 + n_1$, that is:

- If $c_2 = k < \ell < c_1$, $v_{i+1}$ is reached by $s$-paths of composition $(\ell - 1, 1)$ and $I_R(v_{i+1}) = \ell$ because $(A_1^{\ell-2, 1, 0}A_2)B_1 = y_{i+1}$.
- If $c_1 < \ell < c_2$, no vertices are reached by $s$-paths of composition $(\ell - 1, 1)$ since $(A_1^{\ell-2, 1, 0}A_2)B_1 = y_{i+1} + y_{i+2}$.
- If $c_2 < \ell < k + n_1$, the vertex $v_{i+1}$ is already reached by $s$-paths of composition $(\ell - 1, 0)$ (see (7)).

Therefore, to conclude the study of 1-cycle, it is necessary to analyze when the vertex $v_{i+1}$ with $c_1 < \ell \leq c_2$ is reached and hence its local reachability index $I_R(v_{i+1})$ may be derived.

Note that for every $v_{i+1}$, with $c_1 < \ell < c_2$, there exists an $s$-path of composition $(\ell - 1, 1)$ ending in it. However, this vertex is not reached by this $s$-path since there exists another one of the same composition ending in $w_{i+1}$. This last statement is due to the fact that an $s$-path of composition $(\ell - 1, 1)$ (with length $\ell$) ends in $w_{i+1}$ if and only if $h + 1 \leq \ell \leq h + n_2$ which is always true.

On the whole, the $s$-paths ending in $v_{i+1}$, with $c_1 < \ell < c_2$, have a composition $(\ell - 1 + r(n_1 - 1), 1 + r)$, for each $r \in \mathbb{Z}_+$. In addition, taking into account that the length of such $s$-paths is $\ell + r(m_1)$, there exists another $s$-path with the same composition ending in $w_{i+1}$ if and only if

$$h + 1 + r(m_1) \leq \ell + r(m_1) \leq h + (r + 1)n_2. \quad (9)$$

The inequality (9-(b)) is true for all $r$ since $\ell + m_1 \leq c_1 + m_1 = \min \{k, h\} + n_1 + n_2 \leq h + (r + 1)n_1$ or $h + (r + 1)n_2$, while (9-(a)) is true if and only if $r \leq (h - \ell - 1)/(n_2 - n_1)$.

Thus, there exist no $s$-paths of composition $(\ell - 1 + r(n_1 - 1), 1 + r)$ ending in a vertex in 2-cycle if $r \in \mathbb{Z}_+$ and $r > (h - \ell - 1)/(n_2 - n_1)$. Then, the vertex $v_{i+1}$ is reached using an $s$-path of composition $(\ell - 1 + r(n_1 - 1), 1 + r)^\ell$ with $r = \lfloor (\ell - h - 1)/(n_2 - n_1) \rfloor + 1$. Therefore,

$$I_R(v_{i+1}) = \ell + n_1 \left( \frac{\ell - h - 1}{n_2 - n_1} + 1 \right). \quad (10)$$

Note that every vertex in 1-cycle has been reached and its associated index calculated. The previous steps are summarized in the following table:
Finally, it is obvious that the highest value of (10) is obtained when \( \ell = c_2 = \min(k, h) + n_1 \). Then, \( |\epsilon| = \min(k, h) \) and
\[
I_R(\epsilon_{\min(k, h)}) = \min\{k, h\} + n_1 \left[ \frac{\min(k, h) + n_2 - h - 1}{n_2 - n_1} + 1 \right]. \tag{11}
\]

Secondly, let us study the vertices of the second cycle reached by \( s \)-paths of composition \((\ell - 1, 1)\).

- If \( c_1 \leq \ell \leq c_2 \), the vertex \( w_{(1)} \) is reached by \( s \)-paths of composition \((\ell - 1, 1)\) and \( I_R(w_{(1)}) = \ell \) because \((A_1^{-(\ell-2)}A_2^1)B_1 = y_{w_{(1)}}\).

- If \( c_1 < \ell < c_2 \), no vertices are reached by \( s \)-paths of composition \((\ell - 1, 1)\) because \((A_1^{-(\ell-2)}A_2^1)B_1 \neq y_{w_{(1)}}\).

- If \( c_2 < \ell < c_1 + n_2 \), the vertex \( w_{(2)} \) is reached and \( I_R(w_{(2)}) = \ell \) since \((A_1^{-(\ell-2)}A_2^1)B_1 = y_{w_{(2)}}\).

Therefore, to conclude the study of 2-cycle, it is necessary to analyze when the vertex \( w_{(1)} \) with \( c_1 < \ell \leq c_2 \) is reached and hence its local reachability index \( I_R(w_{(1)}) \) may be derived.

In general, the \( s \)-paths ending in \( w_{(1)} \) have a composition \((\ell - 1 + r(n_1 - 1), 1 + r)\) for every \( r \in \mathbb{Z}_+ \). Moreover, taking into consideration that the length of these \( s \)-paths is \( \ell + r(n_2) \), there exists another \( s \)-path with the same composition ending in \( v_{(r+1)} \) if and only if
\[
k + 1 + r(n_2) \leq \ell + r(n_2) + 1 \leq (r + 1)n_1. \tag{12}
\]
The inequality (12) is true for all \( r \) since \( \ell > k \) and \( n_1 < n_2 \), while (12-\( b \)) is true if and only if \( r \leq \frac{(k + n_1 - \ell)}{(n_2 - n_1)} \).

Thus, there exist no \( s \)-paths of composition \((\ell - 1 + r(n_2 - 1), 1 + r)\) ending in a vertex in 1-cycle if \( r \in \mathbb{Z}_+ \) and \( r > \frac{(k + n_1 - \ell)}{(n_2 - n_1)} \). Then, the vertex \( w_{(1)} \) is reached using an \( s \)-path of composition \((\ell - 1 + r(n_2 - 1), 1 + r)\) with \( r = \frac{(k + n_1 - \ell)}{(n_2 - n_1)} + 1 \). Therefore,
\[
I_R(w_{(1)}) = \ell + n_2 \left[ \frac{k + n_1 - \ell}{n_2 - n_1} + 1 \right]. \tag{13}
\]

Note that every vertex in 2-cycle has been reached and its associated index calculated which is summarized as follows:

<table>
<thead>
<tr>
<th>Second Cycle</th>
<th>Case</th>
<th>Composition</th>
<th>( I_R(w_{(1)}) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( h &lt; \ell \leq c_1 )</td>
<td>((\ell - 1, 1))</td>
<td>\ell</td>
<td></td>
</tr>
<tr>
<td>( c_1 &lt; \ell \leq c_2 )</td>
<td>((\ell - 1 + r(n_2 - 1), 1 + r))</td>
<td>( \ell + n_2 )</td>
<td></td>
</tr>
<tr>
<td>( c_1 + n_2 &lt; \ell \leq h + n_2 )</td>
<td>((\ell - 1, 1))</td>
<td>\ell</td>
<td></td>
</tr>
<tr>
<td>( c_1 + n_2 &lt; \ell \leq h + n_2 )</td>
<td>((\ell - 1, 1))</td>
<td>\ell</td>
<td></td>
</tr>
</tbody>
</table>

Following, it is analyzed which is the vertex \( w_{(2)} \) with \( c_1 < \ell \leq c_2 \) to provide the maximum local reachability index of the vertices in 2-cycle and for that, it is shown which is the maximum \( \ell \in \mathbb{N} \) with \( c_1 < \ell \leq c_2 \) leading to the maximum length of \( s \)-paths reaching \( w_{(2)} \).

To simplify, let us consider \( r_1 := \frac{k + n_1 - \ell}{n_2 - n_1} \). Obviously, the highest value \( \tilde{r} \) of \( r_1 \) is obtained when \( \ell = c_1 + 1 \) as \( c_1 < \ell \leq c_2 \). Therefore, \( \tilde{r} := r_{c_1 + 1} \).

Let us define
\[
\tilde{\ell} := n_1 + k - (n_2 - n_1)\tilde{r}. \tag{14}
\]

Thus, \( \tilde{\ell} \leq n_1 + k \) and \( c_1 < \tilde{\ell} \leq c_2 \) because \( h + n_1 - 1 \) and
\[
\tilde{\ell} \leq n_1 + h + (n_1 - 1) - (n_2 - n_1) \left( \frac{(n_1 - 1) - (n_2 - n_1)}{n_2 - n_1} \right) = n_1 + h + (n_1 - 1) \mod (n_2 - n_1) \leq h + n_2 - 1.
\]

In addition, \( r_1 = \frac{n_1 - k - \tilde{r}}{n_2 - n_1} = \frac{n_1 - k - \tilde{\ell}}{n_2 - n_1} \).

Therefore,
\[
I_R(w_{(2)}) = k + n_1 \left[ \frac{k + n_2 - \max\{k, h\} - 1}{n_2 - n_1} \right] + n_2, \tag{15}
\]

and \( I_R(w_{(2)}) \geq I_R(w_{(1)}) \) for all \( \ell, c_1 < \ell \leq \tilde{\ell} \).

Lastly, when \( \ell > \tilde{\ell} \), \( \ell = \tilde{\ell} + t(n_2 - n_1) + a \) and \( a := (\ell - \tilde{\ell}) \mod (n_2 - n_1) \).

Hence,
\[
I_R(w_{(2)}) = I_R(w_{(1)}), \tag{16}
\]

Example 3: Let \((A_1, A_2, B_1, B_2)\) be a positive 2-D system given by
\[
A_1 = \begin{bmatrix} e_2 & e_3 & e_4 & 0 & 0 & e_6 & e_7 & e_8 & 0 & e_5 \end{bmatrix} \in \mathbb{R}_{++}^{9},
A_2 = \begin{bmatrix} 0 & 0 & 0 & e_1 & 0 & 0 & 0 & e_0 \end{bmatrix} \in \mathbb{R}_{++}^{8},
B_1 = \begin{bmatrix} e_1 & e_5 \end{bmatrix} \in \mathbb{R}_{++}^{9}, \quad B_2 = 0 \in \mathbb{R}_{++}^{9},
\]
with 2-D influence digraph \((4, 4); 5, 4\) corresponding to Fig. 3.

Therefore, \( c_1 = \max\{k, h\} = 4 \) and \( c_2 = \min\{k + n_1, h + n_2\} = 9 \). Hence, following the case \( c_1 < c_2 \) it is obtained
\[
I_R(v_1) = 9, \quad I_R(v_2) = 14, \quad I_R(v_3) = 19, \quad I_R(v_4) = 24.
\]

Thus, the local reachability index is
\[
I_{LR} = k + n_1 \left[ \frac{k + n_2 - \max\{k, h\} - 1}{n_2 - n_1} \right] + n_2 = 4 + 4 \left[ \frac{4 + 5 - 4 - 1}{5 - 4} \right] + 5 = 25 = I_R(w_5).
\]

Finally, let us verify that the above formula obtained in the case \( c_1 < c_2 \) is also useful for the first case \( c_1 \geq c_2 \).

As we have seen, when \( c_1 \geq c_2 \) then \( h = \max\{k, h\} \geq k + n_1 \). Consequently,
\[
k + n_2 - \max\{k, h\} - 1 = k + n_2 - h - 1 = (k + n_1 - h) + (n_2 - n_1 - 1) \leq n_2 - n_1 - 1.
\]

Hence, \( k + n_2 - \max\{k, h\} - 1 \) is reduced to \( I_{LR} = k + n_2 \), which is the desired aim. Summing-up, the local reachability index of the system with 2-D influence digraph \((n_1, \{k\}, n_2, \{h\})\) is given in all cases by expression (16).
Afterwards, the following result holds.

**Theorem 1:** Let \((A_1, A_2, B_1, B_2)\) be a positive 2-D system with 2-D influence digraph \((n_1, \{k\}; n_2, \{h\})\). Then, this system is reachable if and only if \(n_1 \neq n_2\) and so the local reachability index \(I_{LR}\) is given by

\[
I_{LR} = \frac{k + \tilde{n}_1 - \max\{k, h\} - 1}{\tilde{n}_2 - \tilde{n}_1} + \tilde{n}_2,
\]

where \(\tilde{n}_1 = \min\{n_1, n_2\}\) and \(\tilde{n}_2 = \max\{n_1, n_2\}\).

**Proof:** If \(n_1 = n_2\), Lemma 1 establishes that the system is not reachable. For the case \(n_1 < n_2\), the local reachability index of the system \((n_1, \{k\}; n_2, \{h\})\) is given by expression (16) as it has been analyzed formerly. Finally, the case \(n_1 > n_2\) is reduced to the previous case taking the system \((A_2, A_1, B_1, B_2)\) with 2-D influence digraph \((n_2, \{h\}; n_1, \{k\})\).

Note that for this class of special systems the local reachability index always appears to be associated with the vertices of the second cycle \(v_j|v_n\), being \(j\) given by expression (14). Besides that, if \(h \leq k\), the value of \(h\) does not influence the local reachability index.

**Lemma 2:** Let \((A_1, A_2, B_1, B_2)\) be a positive 2-D system with 2-D influence digraph \((n_1, \{k\}; n_2, \{h\})\), being \(n_1\) and \(n_2\) fixed natural numbers with \(n_1 < n_2\) and \(n_1 + n_2 = n\). Then, the maximum local reachability index is achieved by those systems satisfying \(k = n_1\) and \(h \leq k\). Moreover, under these circumstances,

\[
I_{LR} = n_1 + \frac{n_2 - 1}{n_2 - n_1}.
\]

**Proof:** Since \(k + n_2 - \max\{k, h\} - 1\) is greatest when \(\max\{k, h\} = k\) and when so, \(k + n_2 - \max\{k, h\} - 1 = n_2 - 1\). Then by \(k \leq n_1\) and \(n_1 + n_2 = n\), the results are held.

**Theorem 2:** Any systems \((A_1, A_2, B_1, B_2)\) with 2-D influence digraph \((n_1, \{k\}; n_2, \{h\})\) where \(n_1 \neq n_2\) and \(n_1 + n_2\) satisfy the condition that its local reachability index \(I_{LR}\) is upper bounded by \(\frac{(n + 1)^2}{4}\). In addition, for every odd natural number \(n\) there exists at least one system with 2-D influence digraph \((n_1, \{k\}; n_2, \{h\})\) such that its local reachability index is equal to this upper bound.

**Proof:** The maximum value of (17) occurs if \(n_2 - n_1 = 1\). In this case \(n = n_1 + n_2 = 2n_1 + 1\) is an odd natural number and \(n_1 = \frac{n - 1}{2}\), \(n_2 = \frac{(n + 1)}{2}\). Then,

\[
I_{LR} = n_1 + \frac{n_2 - 1}{n_2 - n_1} = n_1 + \frac{4}{4} = n + \frac{n - 1}{4} = \frac{(n + 1)^2}{4}.
\]

If \(n\) is an even natural number, the minimum value of \(n_2 - n_1 = 2\) and it occurs when \(n_1 = \frac{n}{2} - 1\) and \(n_2 = \frac{n}{2} + 1\). In this case,

\[
I_{LR} = n_1 + \frac{n_2 - 1}{n_2 - n_1} = n + \frac{n}{2} - 1 = \frac{n}{4}.
\]

Notice that \(n = n_1 + n_2 \geq 3\) as \(n_1 \neq n_2\) and if \(n\) is an even natural number then \(n \geq 4\). Hence, if \(n\) is a multiple of 4 i.e. \(n = 4p, p \in \mathbb{N}\), then \(\frac{n}{4} = p = \frac{n}{4}\) and so (18) is equal to \(\frac{(n + 1)^2}{8}\). Finally, if \(n\) is even but not multiple of 4 that is \(n = 4p + 2\), \(p \in \mathbb{N}\), then \(\frac{n}{4} = \frac{n}{8} = \frac{p + 2}{2}\) and so (18) is equal to \(\frac{(p + 1)^2}{8}\). To shorten,

\[
\max\{I_{LR}\} = \begin{cases} 
\frac{(n + 1)^2}{4}, & \text{if } n \text{ is an odd number}, \\
\frac{n^2 + 6n}{8}, & \text{if } n \text{ is a multiple of 4 and} \\
\frac{(n + 2)^2}{8}, & \text{otherwise}.
\end{cases}
\]

It is obvious that \(\frac{(n + 2)^2}{8} < \frac{(n + 1)^2}{4} < \frac{(n + 2)^2}{8} < \frac{(n + 1)^2}{4}\), \(\forall n \geq 2\). Hence, \(I_{LR} \leq \frac{(n + 1)^2}{4}\), which completes the proof.

**IV. REMARKS ON THE LOCAL REACHABILITY INDEX**

Let us see that the upper bound obtained in this paper is only valid for the class of the chosen systems. There are many examples whose local reachability index is greater than \(\frac{(n + 1)^2}{4}\). In the same way, such examples show us that all well-known conjectures are not valid for any positive 2-D systems.

**Example 4:** Let the system \((A_1, A_2, B_1, B_2)\) with 2-D influence digraph corresponding to Fig. 4, note that \(I_R(v_1) = 4\).

**Fig. 4.** Digraph for system in example 4.

1. \(I_R(v_2) = 2\) and \(I_R(v_3) = 6\) since all \(s\)-paths of composition \((1, 0), (2, 0)\) and \((3, 3)\) end in the single vertex \(v_1, v_2\) and \(v_3\), respectively. Thus, the system is locally reachable and \(I_{LR} = 6\). Therefore, the upper bound \(\frac{(n + 1)^2}{4}\) given in the precedent section is not useful for this system since \(I_{LR} = 6 > 4 = \frac{(n + 1)^2}{4}\) if \(n = 3\).

Therefore, an upper bound of the local reachability index of any positive 2-D system must be greater than \(\frac{(n + 1)^2}{4}\), which shows that the conjecture given in [1] fails.

**REFERENCES**


