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Schur complement of general H-matrices

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SUMMARY

It is well-known that the Schur complement of some H-matrices is an H-matrix. In this paper, the Schur complement of any general H-matrix is studied. In particular it is proved that the Schur complement, if it exists, is an H-matrix and it is studied to which class of H-matrix the Schur complement belongs. In addition, results are given for singular irreducible H-matrices and for the Schur complement of nonsingular irreducible H-matrices. Copyright © 2000 John Wiley & Sons, Ltd.

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1. Introduction

M-matrices and the more general class of H-matrices have been applied in different problems of mathematics and other sciences. One of the most important applications of these kind of matrices is in Numerical Linear Algebra; more concretely in the solution of linear systems by the LU factorization and by the Schur complement as well as in the construction of preconditioners.

The concept of nonsingular M-matrix and H-matrix was introduced by Ostrowski [14] in the study of the convergence of iteration processes and spectral theory. Later, these definitions were extended by Fiedler and Ptak to possible singular M-matrices [6] and H-matrices [7]. Moreover, the study of nonsingular or general M-matrices and H-matrices was widely extended (see [19], [2] and the references therein).

In [5] Ky Fan stated that the Schur complement, with respect to a principal submatrix of size 1 of a nonsingular M-matrix, is a nonsingular M-matrix. This result was extended to general Schur complements by Crabtree [4] (see also [22]) and for nonsingular H-matrices by Polman [15]. Also, from the results of Johnson [12] and Smith [18] the irreducibility of the Schur complement of general M-matrices with respect to an invertible principal submatrix was characterized (see also [21]). Further, in the book [17] there is a detailed study of different classes of matrices which are closed for the Schur complement, in particular the invertible class of H-matrices.

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In this paper we deal with general M-matrices, that is matrices that can be written as $A = rI - B$, where $r \geq \rho(B)$ and B is a nonnegative matrix, these matrices can be singular, and with general H-matrices, that is, matrices whose comparison matrix $\mathcal{M}(A)$ is a general M-matrix, see [2]. We denote by $|A|$ the matrix whose entries are $|a_{ij}|$, for $i, j = 1, \ldots, n$. We say that two matrices A and B are equimodular if $|A| = |B|$, clearly A and $\mathcal{M}(A)$ are equimodular.

In [3] a partition of the set of all H-matrices in three classes is given: the *invertible* class $\mathcal{H}_I$ containing all H-matrices such that their comparison matrix is a nonsingular M-matrix, all matrices in this class are nonsingular; the *mixed* class $\mathcal{H}_M$ formed for all H-matrices having singular comparison matrix but with at least one equimodular matrix being nonsingular; and the *singular* class $\mathcal{H}_S$ with all H-matrices such that all their equimodular matrices are singular, all matrices in this class are reducible.

The goal of this paper is to establish that the Schur complement of a general H-matrix, if it exists, is also an H-matrix. Moreover we establish that the Schur complements belong to the same class of H-matrices together with the original matrix A if A belongs to $\mathcal{H}_I$ or $\mathcal{H}_S$ or if A is a singular matrix in $\mathcal{H}_M$. For nonsingular matrices in $\mathcal{H}_M$ we give some conditions on the graph of A to determine when their Schur complements remain in $\mathcal{H}_M$ or improve to $\mathcal{H}_I$.

The background on Schur complements of general M-matrices and H-matrices in $\mathcal{H}_I$ is contained in section 2, where we include some results that will be used later. The Schur complement of H-matrices in $\mathcal{H}_M$ and $\mathcal{H}_S$ are studied in sections 3 and 4 respectively. The paper ends gathering the main results.

2. Background and Schur complements on $\mathcal{H}_I$

Let $A \in \mathbb{C}^{n\times n}$ and let $\alpha, \beta \subseteq \langle n\rangle = \{1, 2, \ldots, n\}$. As usual, $A(\alpha, \beta)$ denotes the submatrix of A with row and column indices all those in α and β, respectively, and $A(\alpha)$ denotes the principal submatrix of A with row and column indices in α. Throughout the paper, we denote the cardinality of the set α by $\operatorname{card}(\alpha)$ and the complementary subset of α in $\langle n\rangle$ by α'. Finally, we write the strict set inclusion by $X \subset Y$.

Given a nonsingular proper principal submatrix $A(\alpha)$, its Schur complement with respect to $A(\alpha)$ is denoted by $S_\alpha(A)$. That is, if $\operatorname{card}(\alpha) = k$, then $S_\alpha(A)$ is the $(n-k) \times (n-k)$ matrix

$$S_\alpha(A) = A(\alpha') - A(\alpha', \alpha)A(\alpha)^{-1}A(\alpha, \alpha').$$

Recall that properties of H-matrices are closely related with generalized diagonal dominance. In order to maintain these properties, we should restrict our study to Schur complements with respect to principal submatrices. In this sense we have to consider symmetric permutations of A. Let P be the permutation matrix such that

$$PAP^T = \begin{bmatrix} A(\alpha) & A(\alpha, \alpha') \\ A(\alpha', \alpha) & A(\alpha') \end{bmatrix},$$

then, the Schur complement is part of the block LU factorization

$$PAP^T = LU = \begin{bmatrix} I & 0 \\ A(\alpha', \alpha)A(\alpha)^{-1} & I \end{bmatrix} \begin{bmatrix} A(\alpha) & A(\alpha, \alpha') \\ 0 & S_\alpha(A) \end{bmatrix}$$

and $\det(A) = \det(P^TAP) = \det(A(\alpha))\det(S_\alpha(A))$. Then A is nonsingular if and only if $S_\alpha(A)$ is nonsingular.

The results given in [5, Lemma 1] and [4, Lemma 1] for nonsingular M-matrices, can be easily generalized to any M-matrix as follows.

Theorem 1. *Let A be an M-matrix and let $\alpha \subset \langle n \rangle$ such that $A(\alpha)$ is nonsingular, then $S_\alpha(A)$ is an M-matrix.*

To study the Schur complement of general H-matrices we need the interesting inequality obtained in [1, Lemma 1] and, with some different conditions, in [13, Theorem 1]. This inequality also holds for singular H-matrices.

Lemma 1. *Let A be an H-matrix and let $\alpha \subset \langle n \rangle$ such that $A(\alpha) \in \mathcal{H}_I$, then*

$$S_\alpha(\mathcal{M}(A)) \leq \mathcal{M}(S_\alpha(A)). \tag{1}$$

and $S_\alpha(A)$ is an H-matrix.

Proof. Since $\mathcal{M}(A)(\alpha) = \mathcal{M}(A(\alpha))$ is a nonsingular M-matrix and $A(\alpha)$ is nonsingular, both Schur complements can be computed.

The proof of the inequality (1) follows the steps of [13, Theorem 1] considering that what is really used in the proof is that $[\mathcal{M}(A)(\alpha)]^{-1} \geq 0$. Further, $S_\alpha(\mathcal{M}(A))$ is an M-matrix from Theorem 1, and by inequality (1), $\mathcal{M}(S_\alpha(A))$ is also an M-matrix and so $S_\alpha(A)$ is an H-matrix. □

Lemma 1 can be stated for H-matrices in $\mathcal{H}_I$ as follows.

Corollary 1 ([15, Lemma 3]) *Let $A \in \mathcal{H}_I$ and let $\alpha \subset \langle n \rangle$, then $S_\alpha(A) \in \mathcal{H}_I$.*

Even if Lemma 1 applies to general H-matrices, the generalization is not complete since there may be H-matrices with nonsingular principal submatrix $A(\alpha)$ but with singular $\mathcal{M}(A(\alpha))$. Then $S_\alpha(A)$ exists but we cannot apply Lemma 1 to conclude that $S_\alpha(A)$ is an H-matrix. However, if the matrix is an irreducible M-matrix we obtain the following result.

Theorem 2. *Let A be an irreducible M-matrix and let $\alpha \subset \langle n \rangle$, then the Schur complement $S_\alpha(A)$ exists and it is an M-matrix. Moreover, $S_\alpha(A)$ is irreducible unless A is singular and* $\operatorname{card}(\alpha) = n-1$.

Proof. Any principal submatrix of an irreducible M-matrix is a nonsingular M-matrix. Then, $A(\alpha)$ is nonsingular and $S_\alpha(A)$ is an M-matrix. If A is singular and $\operatorname{card}(\alpha) = n-1$, then $S_\alpha(A) = [0]$ †. Otherwise, $S_\alpha(A)$ is irreducible ([18, Lemma 2.1 (ii)]). □

Theorem 2 cannot be stated for H-matrices, as seen in the following example.

Example 1. *Consider the irreducible matrix $A \in \mathcal{H}_I$*

$$A = \begin{bmatrix} 3 & -1 & -1 \\ 1 & 3 & 1 \\ -1 & \frac{1}{3} & 3 \end{bmatrix}$$

Taking $\alpha = \{1\}$ the Schur complement is the reducible matrix

$$S_{\{1\}}(A) = \begin{bmatrix} 10/3 & 4/3 \\ 0 & 8/3 \end{bmatrix}.$$

† As usual, the 1×1 null matrix is considered reducible.

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3. Schur complements in $\mathcal{H}_M$

Note that any matrix $A \in \mathcal{H}_M$, singular or not, has nonzero diagonal elements, its comparison matrix $\mathcal{M}(A)$ is singular but at least one equimodular matrix is nonsingular (see [3] for details). The matrix A may be irreducible or not; thus we can identify two cases.

3.1. Irreducible case

We prove that in this case the Schur complement always exists and is an H-matrix (related results are given in [8, 9, 1]). We recall that if A is a (nonsingular) M-matrix, any principal submatrix $A(\alpha)$ is a (nonsingular) M-matrix, see [2]. The same happens for H-matrices, but the point is that a principal submatrix of an H-matrix in $\mathcal{H}_M$ or in $\mathcal{H}_S$ can belong to a different class than the original H-matrix. In the following theorem we deal with this point in the mixed irreducible case.

Theorem 3. *Let $A \in \mathcal{H}_M$ be an irreducible matrix. Then the Schur complement with respect to any $\alpha \subset \langle n \rangle$ exists and $S_\alpha(A)$ is an H-matrix.*
Moreover,

1. *$S_\alpha(A) \in \mathcal{H}_I$ (and A is nonsingular) if $S_\alpha(\mathcal{M}(A)) < \mathcal{M}(S_\alpha(A))$.*
2. *$S_\alpha(A) = [0] \in \mathcal{H}_S$ and it is reducible if A is singular and $\mathrm{card}(\alpha) = n - 1$. In this case $S_\alpha(\mathcal{M}(A)) = \mathcal{M}(S_\alpha(A))$.*
3. *Otherwise, $S_\alpha(A) \in \mathcal{H}_M$ is an irreducible matrix (singular or not).*

Proof. Since $\mathcal{M}(A)$ is an irreducible M-matrix, by Theorem 2 we can compute the Schur complement $S_\alpha(\mathcal{M}(A))$ with respect to any proper principal submatrix. Since $\mathcal{M}(A)$ is singular, then $S_\alpha(\mathcal{M}(A))$ is a singular M-matrix. Moreover, as all proper principal submatrices of $\mathcal{M}(A)$ are nonsingular M-matrices, all proper principal submatrices of A are in $\mathcal{H}_I$, and then by Lemma 1 the inequality (1) holds and $S_\alpha(A)$ is an H-matrix.

To end the proof we distinguish three cases.

Case 1. Consider $S_\alpha(\mathcal{M}(A)) < \mathcal{M}(S_\alpha(A))$. If $S_\alpha(A)$ is irreducible‡, then $\mathcal{M}(S_\alpha(A))$ is an irreducible Z-matrix greater than the M-matrix $S_\alpha(\mathcal{M}(A))$. Then, $\mathcal{M}(S_\alpha(A))$ is a nonsingular M-matrix and consequently $S_\alpha(A) \in \mathcal{H}_I$, so A is nonsingular.

If $S_\alpha(A)$ is reducible and $S_\alpha(\mathcal{M}(A)) < \mathcal{M}(S_\alpha(A))$, since both matrices are M-matrices we can state

$$S_\alpha(\mathcal{M}(A)) = rI - B < rI - C = \mathcal{M}(S_\alpha(A))$$

where $r \geq \max_{i \in \langle n \rangle} c_{ii}$, and $0 \leq C < B$. Recall that, as we said before $S_\alpha(\mathcal{M}(A))$ is singular and irreducible, then by [20, Corollary 2.5] we have $r = \rho(B) > 0$. Since B is irreducible and C is reducible, we can assume, without lost of generality that

$$C = \begin{bmatrix} C_{11} & C_{12} & \cdots & C_{1k} \\ 0 & C_{22} & \cdots & C_{2k} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & C_{kk} \end{bmatrix} < B = \begin{bmatrix} B_{11} & B_{12} & \cdots & B_{1k} \\ B_{21} & B_{22} & \cdots & B_{2k} \\ \vdots & \vdots & \ddots & \vdots \\ B_{k1} & B_{k2} & \dots & B_{kk} \end{bmatrix}$$

‡Note that the case $S_\alpha(\mathcal{M}(A)) = [0] < \mathcal{M}(S_\alpha(A))$ is included here.

where all diagonal blocks C_{ii} are irreducible or the 1×1 null matrix. The case $C_{ii} = [0]$ for all $i = 1, 2, \dots, k$, with $k = n$, implies $\rho(C) = 0$ and so $r > \rho(C)$. Then $\mathcal{M}(S_\alpha(A))$ is a nonsingular M-matrix and so $S_\alpha(A) \in \mathcal{H}_I$.

Otherwise, let C_{ii} be the irreducible diagonal block with larger spectral radius, so $\rho(C) = \rho(C_{ii})$. The corresponding block B_{ii} is a principal submatrix of the irreducible nonnegative matrix B, then applying [20, Lemma 2.6] $\rho(B_{ii}) < \rho(B)$. Thus

$$\rho(C) = \rho(C_{ii}) \leq \rho(B_{ii}) < \rho(B) = r$$

then $\mathcal{M}(S_\alpha(A))$ is a nonsingular M-matrix, and $S_\alpha(A) \in \mathcal{H}_I$ and, so, A is nonsingular.

Case 2. It is straightforward.

Case 3. As $S_\alpha(\mathcal{M}(A))$ is a singular and irreducible M-matrix, and, since $S_\alpha(\mathcal{M}(A)) = \mathcal{M}(S_\alpha(A))$, we see that $S_\alpha(A)$ is an H-matrix, but $S_\alpha(A) \notin \mathcal{H}_I$. By irreducibility, $S_\alpha(\mathcal{M}(A))$ is not in $\mathcal{H}_S$ (see [3]). Then $S_\alpha(A)$ remains in $\mathcal{H}_M$. □

Observe from the last theorem that any Schur complement of an H-matrix in $\mathcal{H}_M$ does not decrease the quality of the original matrix, that is, the Schur complement remains in $\mathcal{H}_M$ or improves to $\mathcal{H}_I$, unless A is singular and $\operatorname{card}(\alpha) = n-1$. In this special case one obtains the null matrix which belongs to $\mathcal{H}_S$. (Note that in the set of 1×1 matrices, there are only two classes of H-matrices, $\mathcal{H}_I$ and $\mathcal{H}_S$.)

Moreover, the case 2 of Theorem 3 is not the only situation in which the Schur complement of an irreducible matrix in $\mathcal{H}_M$ can be reducible, as the following example shows. Recall that Example 1 illustrates an analogous situation for a matrix in $\mathcal{H}_I$. Then, while the property of being an H-matrix is inherited by Schur complements, irreducibility is only inherited by Schur complements of M-matrices (see Theorem 2).

Example 2. *Consider the irreducible matrix $B \in \mathcal{H}_M$*

$$B = \begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 2 & 1 & 3 \end{bmatrix}.$$

For $\alpha = \{1\}$, one obtains

$$S_{\{1\}}(B) = \begin{bmatrix} 3/2 & 1/2 \\ 0 & 2 \end{bmatrix}$$

which is a reducible matrix in $\mathcal{H}_I$.

The above example illustrates the following conclusion, according to Theorem 2 and Case 1 of Theorem 3.

Corollary 2. *Let A be an irreducible matrix in $\mathcal{H}_M$. If $S_\alpha(A) \neq [0]$ is reducible, then $S_\alpha(A) \in \mathcal{H}_I$.*

Theorem 3 gives the following characterization of singular irreducible H-matrices in $\mathcal{H}_M$.

Corollary 3. *Let A be an irreducible matrix in $\mathcal{H}_M$. The matrix A is singular if and only if $S_\alpha(\mathcal{M}(A)) = \mathcal{M}(S_\alpha(A))$ holds for all $\alpha \subset \langle n \rangle$.*

Proof. The if part is obtained choosing α such that $\operatorname{card}(\alpha) = n - 1$, and recalling that in this case $S_\alpha(\mathcal{M}(A)) = [0]$, and so A is singular. The only if part is just part 1 of Theorem 3. □

Singular irreducible H-matrices can be characterized as well using their comparison matrix. In particular, when the H-matrix has positive diagonal elements, we have the following result in collaboration with Schneider [16].

Theorem 4. *Let A be a singular irreducible H-matrix with positive diagonal entries. Then, A is diagonally similar to its comparison matrix.*

Proof. Let $D = \mathrm{diag}(h_{11}, h_{22}, \ldots, h_{nn})$, then the matrix $H_1 = D^{-1}A$ is a singular irreducible H-matrix with all diagonal entries equal to 1.

Let us write $H_1 = I + K$, since H_1 is singular, then $-1 \in \sigma(K)$. On the other hand $\mathcal{M}(H_1) = M_1 = I - |K|$ is a singular M-matrix. Then $\rho(|K|) = 1$. By [10, Theorem 8.1.18] $\rho(K) \leq \rho(|K|) = 1$. Then, -1 is the maximum eigenvalue of K in absolute value, so $\rho(K) = 1$. Now, taking $K = B$ and $|K| = A$ in Wielandt's Theorem [10, Theorem 8.4.5], there exists a diagonal unitary matrix D_1 such that

$$K = -1 D_1 |K| D_1^{-1}$$

then, $H_1 = I + K = I - D_1|K|D_1^{-1} = D_1 M_1 D_1^{-1}$. Finally, we recover the matrix A as

$$A = DH_1 = DD_1M_1D_1^{-1} = D_1DM_1D_1^{-1} = D_1MD_1^{-1}$$

where $M = \mathcal{M}(A)$. □

In addition, we can give the general characterization of singular and irreducible H-matrices in $\mathcal{H}_M$. Another proof is given by Johnson [11].

Corollary 4. *Let A be an irreducible H-matrix in $\mathcal{H}_M$. Then, A is singular if and only if A is diagonally equivalent to its comparison matrix, that is, if there exist unitary diagonal matrices D_1 and D_2 such that*

$$A = D_1\mathcal{M}(A)D_2.$$

Proof. Let $H = D_A^{-1}A$, where $D_A = \mathrm{diag}(A)$. Now, applying Theorem 4 to matrix H, we have a diagonal unitary matrix D_1 such that

$$H = D_1\mathcal{M}(H)D_1^{-1}.$$

Then

$$A = D_AH = D_AD_1\mathcal{M}(H)D_1^{-1} = D_AD_1\mathcal{M}(D_A^{-1}A)D_1^{-1} = D_2\mathcal{M}(A)D_3$$

Note that $D_2 = D_AD_1\mathcal{M}(D_A^{-1})$ and $D_3 = D_1^{-1}$ are unitary diagonal matrices. □

It is easy to see from Corollary 3 that all Schur complements of a singular irreducible H-matrix are in $\mathcal{H}_M$, unless the null Schur complement is obtained. Further, these Schur complements are equimodular to the corresponding Schur complement of $\mathcal{M}(A)$ by Corollary 4.

If $A \in \mathcal{H}_M$ is irreducible and nonsingular, then the inequality (1) is strict for at least one α, and the corresponding Schur complements are in $\mathcal{H}_I$. In general it is difficult to say when the strict inequality (1) is reached. We study two different cases: matrices such that the inequality (1) is strict for every $\alpha \subset \langle n \rangle$ (Corollary 5) and matrices such that the strict inequality holds only when $\mathrm{card}(\alpha) = n - 1$ (Theorem 5). Below we give an example of this last case.

Example 3. *Consider the irreducible nonsingular matrix $A \in \mathcal{H}_M$*

$$A = \begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \end{bmatrix}.$$

For all $\alpha \subset \langle n \rangle$ such that $\mathrm{card}(\alpha) < 3$ *the equality $S_\alpha(\mathcal{M}(A)) = \mathcal{M}(S_\alpha(A))$ holds, but* $[0] = S_\alpha(\mathcal{M}(A)) < \mathcal{M}(S_\alpha(A)) = [2]$, *for any α such that* $\mathrm{card}(\alpha) = 3$.

Let A be a square matrix of order $n \geq 3$ such that it can be split as

$$A = D + M, \tag{2}$$

where D is diagonal with nonzero diagonal entries and $|M|$ is a cyclic matrix of index n (see [20]). That is, the graph of M is a cycle of length n.

Theorem 5. *Let $A = D + M$ be a matrix of size $n > 2$, where D is diagonal with nonzero diagonal entries and $|M|$ is a cyclic matrix of index n, and let $\alpha \subset \langle n \rangle$.*

1. *If $A \in \mathcal{H}_M \cup \mathcal{H}_I$ and $\mathrm{card}(\alpha) < n-1$, then $S_\alpha(\mathcal{M}(A)) = \mathcal{M}(S_\alpha(A))$.*
2. *If $A \in \mathcal{H}_M$ is nonsingular and $\mathrm{card}(\alpha) = n-1$, then $[0] = S_\alpha(\mathcal{M}(A)) < \mathcal{M}(S_\alpha(A))$ and $S_\alpha(A) \in \mathcal{H}_I$.*

Proof. 1. We analyze first the case of $\mathrm{card}(\alpha) = 1$.

Without loss of generality suppose that $\gamma = \{1, 2, \ldots, n, 1\}$ is the cycle of the graph of M and that $\alpha = n$. That is, the nonzero entries of A are: those of the main diagonal, the $a_{i,i+1}$ entries for $i = 1, \ldots, n-1$, and the $a_{n,1}$ entry. Then, all entries $\widetilde{a}_{ij}$ of $S_\alpha(A)$, with $i, j \neq n$, remains unchanged except exactly one entry; that is:

- $\widetilde{a}_{i,j} = a_{ij} \quad \text{if} \quad i \neq n-1 \text{ and } j \neq 1$
- $\widetilde{a}_{n-1,1} = 0 - \frac{a_{n-1,n}a_{n,1}}{a_{nn}} \neq 0$

Therefore, the Schur complement can be computed simply by deleting the last row and column of A and updating the entry on its left bottom corner.

Observe that the same happens for the Schur complement of the comparison matrix. Then, if $\widetilde{m}_{ij}$ with $i, j \neq n$ denotes the entries of $S_\alpha(\mathcal{M}(A))$, we have $|\widetilde{m}_{ij}| = |\widetilde{a}_{ij}|$, so $S_\alpha(\mathcal{M}(A)) = \mathcal{M}(S_\alpha(A))$ and the case $\mathrm{card}(\alpha) = 1$ follows.

Note that $S_{\{n\}}(A)$ admits the splitting (2) and so we can work recursively as before until $\mathrm{card}(\alpha) = n-2$, and then the case 1 is completed.

2. When $A \in \mathcal{H}_M$ is nonsingular, since A is irreducible, there exists some α such that $S_\alpha(\mathcal{M}(A)) < \mathcal{M}(S_\alpha(A))$ by case 1, necessarily, $\mathrm{card}(\alpha) = n-1$. □

As it has been used in the previous proof, $S_\alpha(A)$ can be computed recursively by computing the Schur complements with respect to sets of cardinality one. A characterization of $S_\alpha(\mathcal{M}(A)) = \mathcal{M}(S_\alpha(A))$ for these sets, where the matrix A can be irreducible or not, is given in the following result. Recall that the entries of the matrix may be complex, so the sign of the nonzero complex entry a_{ij} is $\mathrm{sign}(a_{ij}) = a_{ij}/|a_{ij}|$.

Theorem 6. *Let $A \in \mathcal{H}_I$ or $A \in \mathcal{H}_M$, and let $\alpha = \{k\}$. Then $S_\alpha(\mathcal{M}(A)) = \mathcal{M}(S_\alpha(A))$ if and only if for all $i, j \neq k$ the following conditions are satisfied*

1. *$i = j$:* $\quad \mathrm{sign}(a_{ii}) = \mathrm{sign}(a_{ik}a_{ki}a_{kk}^{-1}) \quad$ *or* $\quad a_{ik}a_{ki} = 0$
2. *$i \neq j$:* $\quad \mathrm{sign}(a_{ij}) = -\,\mathrm{sign}(a_{ik}a_{kj}a_{kk}^{-1}) \quad$ *or* $\quad a_{ij}a_{ik}a_{kj} = 0$.

Moreover, when $S_\alpha(\mathcal{M}(A)) = \mathcal{M}(S_\alpha(A))$,

$$\mathrm{sign}(\widetilde{a}_{ij}) = \mathrm{sign}(a_{ij}) \qquad \text{if } a_{ij} \neq 0 \text{ and } \widetilde{a}_{ij} \neq 0$$

where $\widetilde{a}_{ij} = a_{ij} - a_{ik}a_{kj}/a_{kk} \in S_\alpha(A), \quad i, j \in \alpha'$.

Proof. Let us denote by m_{ij} and $\widetilde{m}_{ij}$ the entries of $\mathcal{M}(A)$ and $S_\alpha(\mathcal{M}(A))$ respectively. Then $S_\alpha(\mathcal{M}(A)) = \mathcal{M}(S_\alpha(A))$ if and only if $|\widetilde{m}_{ij}| = |\widetilde{a}_{ij}|, \quad i, j \neq k$.

Let us assume first that $S_\alpha(\mathcal{M}(A)) = \mathcal{M}(S_\alpha(A))$ happens.

1. We have to prove that $\operatorname{sign}(a_{ii}) = \operatorname{sign}(a_{ik}a_{ki}a_{kk}^{-1})$ if $a_{ik}a_{ki} \neq 0$. Note that $|\widetilde{a}_{ii}| = \left|a_{ii} - \frac{a_{ik}a_{ki}}{a_{kk}}\right|$. Further, $|\widetilde{m}_{ii}| = \left||a_{ii}| - \frac{|a_{ik}||a_{ki}|}{|a_{kk}|}\right| = |a_{ii}| - \frac{|a_{ik}||a_{ki}|}{|a_{kk}|}$, since $S_\alpha(\mathcal{M}(A))$ is an M-matrix.

In this case, we have $|\widetilde{a}_{ii}| = |\widetilde{m}_{ii}|$, that is,

$$\left|a_{ii} - \frac{a_{ik}a_{ki}}{a_{kk}}\right| = |a_{ii}| - \left|\frac{a_{ik}a_{ki}}{a_{kk}}\right|$$

and then $\operatorname{sign}(a_{ii}) = \operatorname{sign}(a_{ik}a_{ki}a_{kk}^{-1})$.

2. Now we have to prove that $\operatorname{sign}(a_{ij}) = \operatorname{sign}(a_{ik}a_{kj}a_{kk}^{-1})$ if $a_{ij}a_{ik}a_{kj} \neq 0$. In this case we have $|\widetilde{a}_{ij}| = \left|a_{ij} - \frac{a_{ik}a_{kj}}{a_{kk}}\right|$, and $|\widetilde{m}_{ij}| = \left|-|a_{ij}| - \frac{|a_{ik}||a_{kj}|}{|a_{kk}|}\right| = |a_{ij}| + \frac{|a_{ik}||a_{kj}|}{|a_{kk}|} = |a_{ij}| + \left|\frac{a_{ik}a_{kj}}{a_{kk}}\right|$. Now, since $|\widetilde{a}_{ij}| = |\widetilde{m}_{ij}|$, then $\operatorname{sign}(a_{ij}) = -\operatorname{sign}(a_{ik}a_{kj}a_{kk}^{-1})$.

The proof of the converse part is straightforward.

In addition from conditions 1 and 2 it is clear that $\operatorname{sign}(\widetilde{a}_{ij}) = \operatorname{sign}(a_{ij})$ if $a_{ij} \neq 0$ and $\widetilde{a}_{ij} \neq 0$. □

Corollary 5. *Let $A \in \mathcal{H}_M$ be nonsingular such that $a_{ij} \neq 0$ for all $i, j \in \langle n\rangle$. Then $S_\alpha(\mathcal{M}(A)) < \mathcal{M}(S_\alpha(A))$ and $S_\alpha(A) \in \mathcal{H}_I$ for all $\alpha \subset \langle n\rangle$.*

Proof. First we prove that $S_\alpha(A) \in \mathcal{H}_I$ when $\alpha = \{k\}$. Without loss of generality, we assume $k = 1$ and we proceed by contradiction.

Suppose, by Theorem 3, that $S_{\{1\}}(\mathcal{M}(A)) = \mathcal{M}(S_{\{1\}}(A))$. Then, conditions 1 and 2 of Theorem 6 can be written as

$$\frac{\operatorname{sign}(a_{1i})\operatorname{sign}(a_{i1})}{\operatorname{sign}(a_{11})} = \operatorname{sign}(a_{ii}), \qquad \forall i \neq 1,$$

$$\frac{\operatorname{sign}(a_{i1})\operatorname{sign}(a_{1j})}{\operatorname{sign}(a_{11})} = -\operatorname{sign}(a_{ij}), \qquad \forall i, j \neq 1, \quad i \neq j.$$

From these two conditions, we obtain

$$\frac{\operatorname{sign}(a_{1i})\operatorname{sign}(a_{ij})}{\operatorname{sign}(a_{1j})} = -\operatorname{sign}(a_{ii}), \qquad \forall i, j \neq 1, \quad i \neq j. \tag{3}$$

Let $D = \operatorname{diag}(\operatorname{sign}(a_{1i}))$. Since $A = [a_{ij}] = [|a_{ij}|\operatorname{sign}(a_{ij})]$, we have

$$DAD^{-1} = \left[\frac{|a_{ij}|\operatorname{sign}(a_{ij})\operatorname{sign}(a_{1i})}{\operatorname{sign}(a_{1j})}\right] = [p_{ij}].$$

Then using (3) we obtain $p_{ij} = -|a_{ij}|\operatorname{sign}(a_{ii})$, for $i, j \neq 1, i \neq j$. Further the entries of the first row are $p_{1i} = |a_{1i}|\operatorname{sign}(a_{11})$, the entries in the first column are $p_{i1} = |a_{i1}|\operatorname{sign}(a_{ii})$ and the entries on the main diagonal are $p_{ii} = |a_{ii}|\operatorname{sign}(a_{ii}), \forall i$. Therefore, constructing the diagonal matrices

$$D_2 = \operatorname{diag}(-1, 1, 1, \ldots, 1) \quad \text{and} \quad D_3 = \operatorname{diag}(\operatorname{sign}(a_{ii})^{-1})$$

we obtain $D_2D_3DAD^{-1}D_2 = \mathcal{M}(A)$, which is singular and so is A, contradicting the hypothesis.

For a general subset $\alpha \subset \langle n\rangle$, let $k \in \alpha$. Since $S_\alpha(A)$ is a Schur complement of $S_{\{k\}}(A)$ and $S_{\{k\}}(A) \in \mathcal{H}_I$, by Corollary 1, we conclude that $S_\alpha(A) \in \mathcal{H}_I$. □

The converse of this corollary is not true as the following example shows.

Example 4. *The matrix*

$$A = \begin{bmatrix} 1 & -1 & 0 \\ 1 & 2 & 1 \\ 1 & 0 & 1 \end{bmatrix}$$

satisfies $S_\alpha(\mathcal{M}(A)) < \mathcal{M}(S_\alpha(A))$ for all $\alpha \subset \langle n \rangle$, moreover $S_\alpha(A) \in \mathcal{H}_I$ and it is an irreducible nonsingular matrix in $\mathcal{H}_M$, but has some null entries.

Remark. By Theorem 3 and more precisely, by Corollary 3 there is an $\alpha \subset \langle n \rangle$ such that $S_\alpha(\mathcal{M}(A)) < \mathcal{M}(S_\alpha(A))$ holds for irreducible nonsingular H-matrices in $\mathcal{H}_M$, and then $S_\alpha(A) \in \mathcal{H}_I$. This fact assures that based ILU preconditioners for these matrices can be computed breakdown-free.

3.2. Reducible case

Without loss of generality, we can assume that $A \in \mathcal{H}_M$ is already in its normal form

$$A = \begin{bmatrix} A_{11} & A_{12} & \cdots & A_{1p} \\ 0 & A_{22} & \cdots & A_{2p} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & A_{pp} \end{bmatrix} \tag{4}$$

where A_{ii}, for $i = 1, 2, \ldots, p$, are irreducible square H-matrices (see [3, Theorems 5 and 7]), that is, there are no null diagonal blocks and then $A_{ii} \in \mathcal{H}_M$ or $A_{ii} \in \mathcal{H}_I$. Moreover, at least one diagonal block is in $\mathcal{H}_M$, and its comparison matrix is singular. All the properties that we wish to study depend only on these diagonal blocks since off-diagonal blocks do not influence the Schur complement (see [3]). So we reduce our study to the block diagonal submatrices.

To analyze this case we denote by β_i the subset of $\langle n \rangle$ such that $A(\beta_i) = A_{ii}$ ($\cup_{i=1}^{p} \beta_i = \langle n \rangle$, with p that in (4)). Given $\alpha \subset \langle n \rangle$ we denote by $\alpha_i = \alpha \cap \beta_i$, for all $i \in \langle p \rangle$. Consequently, the blocks on the main diagonal of $S_\alpha(A)$ are $S_{\alpha_i}(A_{ii})$, for all $i \in \langle p \rangle$, where if $\alpha_i = \emptyset$, then $S_{\alpha_i}(A_{ii}) = A_{ii}$, while if $\alpha_i = \beta_i$, A_{ii} does not have any influence of $S_\alpha(A)$.

Theorem 7. *Let $A \in \mathcal{H}_M$ be reducible, and let $\alpha \subset \langle n \rangle$. Then,*

1. *The Schur complement of A with respect to $A(\alpha)$ exists if and only if for all i such that $\alpha_i = \beta_i$ the submatrix A_{ii} is nonsingular.*
2. *If the Schur complement $S_\alpha(A)$ exists, it is an H-matrix. In addition,*
   - *(a) If there exists some i such that A_{ii} is singular and $\operatorname{card}(\alpha_i) = \operatorname{card}(\beta_i) - 1$, then $S_\alpha(A) \in \mathcal{H}_S$.*
   - *(b) If $\alpha_i \neq \emptyset$ for any block $A_{ii} = A(\beta_i) \in \mathcal{H}_M$ and the condition $S_{\alpha_i}(A_{ii}) \in \mathcal{H}_I$ holds when $\alpha_i \neq \beta_i$, then $S_\alpha(A) \in \mathcal{H}_I$.*
   - *(c) Otherwise, $S_\alpha(A) \in \mathcal{H}_M$.*

Proof. 1. Since $A(\alpha)$ is a block triangular matrix, it is clear that $A(\alpha)$ is nonsingular if and only if all blocks $A(\alpha_i)$ are nonsingular. If the block $A_{ii} = A(\beta_i)$ is nonsingular it is clear that $A(\alpha_i)$ is nonsingular, therefore the Schur complement of A exists if and only if $A(\alpha_i)$ is nonsingular for every i such that A_{ii} is singular. This last condition holds, if and only if $\alpha_i \neq \beta_i$ since in this case $A(\alpha_i)$ is a proper submatrix of an irreducible H-matrix.

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2. It is clear that the Schur complement is: (i) in $\mathcal{H}_S$ if at least one of its blocks is in $\mathcal{H}_S$, (ii) in $\mathcal{H}_I$ if all its blocks are in $\mathcal{H}_I$, and (iii) in $\mathcal{H}_M$ if none of its blocks is in $\mathcal{H}_S$ and at least one of them is in $\mathcal{H}_M$.

On the other hand, if $A_{ii} \in \mathcal{H}_I$ then $S_{\alpha_i}(A_{ii}) \in \mathcal{H}_I$, and we have to analyze only $S_{\alpha_i}(A_{ii})$ when $A_{ii} \in \mathcal{H}_M$, applying Theorem 3.
(2a) The conditions of this case imply that $S_{\alpha_i}(A_{ii}) \in \mathcal{H}_S$. Hence the Schur complement $S_\alpha(A)$ is in $\mathcal{H}_S$.
(2b) If this condition is satisfied, then the Schur complement $S_{\alpha_i}(A_{ii})$ is in $\mathcal{H}_I$. If all of them are in $\mathcal{H}_I$ then $S_\alpha(A)$ is in $\mathcal{H}_I$. (Note that the blocks corresponding to $\alpha_i = \beta_i$ do not affect the Schur complement).
(2c) If any of the conditions (2a) or (2b) fails, then clearly the Schur complement is in $\mathcal{H}_M$. □

4. Schur complements in $\mathcal{H}_S$

If $A \in \mathcal{H}_S$, then it has some null diagonal entries and it is reducible (see [3, Theorem 7]). Then its normal form has at least one diagonal block which is a 1×1 null matrix. Then the following result is straightforward.

Theorem 8. *Let A be an H-matrix in $\mathcal{H}_S$ and let $\alpha \subset \langle n \rangle$. If the submatrix A_{ii} is nonsingular for all i such that $\alpha_i = \beta_i$, then the Schur complement of A with respect to $A(\alpha)$ exists and $S_\alpha(A) \in \mathcal{H}_S$.*

Note that from the conditions of Theorem 8, null diagonal entries are not in $A(\alpha)$. Then, they are unchanged in the Schur complement.

5. Conclusions

We have proved that the Schur complement of a general H-matrix is also an H-matrix, if it can be computed, which is the case in $\mathcal{H}_I$ or irreducible H-matrices in $\mathcal{H}_M$.

Furthermore, the irreducibility can be lost for nonsingular H-matrices even in $\mathcal{H}_I$. When the Schur complement of an irreducible H-matrix becomes reducible, then it is either in $\mathcal{H}_S$ or in $\mathcal{H}_I$ if A is singular or not, respectively. Moreover, it has been proven that a singular irreducible matrix in $\mathcal{H}_M$ is diagonally equivalent to its comparison matrix, and its Schur complements are equimodular to the corresponding Schur complements of its comparison matrix.

We have also studied the class to which the Schur complement belongs, and the respective classes obtained are shown in Table I. Additionally, for nonsingular irreducible H-matrices in $\mathcal{H}_M$, we have found conditions to guarantee that all Schur complements belong to $\mathcal{H}_I$, and conditions to guarantee that all Schur complements, except for 1×1 Schur complements, remain in $\mathcal{H}_M$. We conjecture that the converse of Theorem 5 is true, but the proof seems to be rather technical.

($\star$) If A is nonsingular or irreducible, then $A(\alpha)$ is always nonsingular. If A is reducible and singular, $A(\alpha)$ is nonsingular if condition 1 in Theorem 7 is fulfilled.

($\heartsuit$) The Schur complement is in $\mathcal{H}_S$ if A is irreducible and $\mathrm{card}(\alpha) = n - 1$, or A is reducible and there exists some i, with A_{ii} singular and $\mathrm{card}(\alpha_i) = \mathrm{card}(\beta_i) - 1$.

In brief, the class of the Schur complement, provided it exists, maintains or improves the initial class of A, except in the case of the Schur complement of a singular (block) matrix is computed with respect to an $(n-1) \times (n-1)$ principal submatrix.

Table I. Summary of the classes.

Class of A	Invertibility of A	$A(\alpha)$	Class of $S_\alpha(A)$
$\mathcal{H}_I$	Nonsingular	Nonsingular	$\mathcal{H}_I$
$\mathcal{H}_M$	Nonsingular	Nonsingular	$\mathcal{H}_M$ or $\mathcal{H}_I$
	Singular	($\star$)	$\mathcal{H}_M$ or $\mathcal{H}_S$ ($\heartsuit$)
$\mathcal{H}_S$	Singular	($\star$)	$\mathcal{H}_S$

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