Is $A \in \mathbb{C}^{n,n}$ a General $H-$Matrix?*

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Abstract

$H-$matrices play an important role in the theory and applications of Numerical Linear Algebra. So, it is very useful to know whether a given matrix $A \in \mathbb{C}^{n,n}$, usually the coefficient of a complex linear system of algebraic equations or of a Linear Complementarity Problem ($A \in \mathbb{R}^{n,n}$, with $a_{ii} > 0$ for $i = 1, 2, \ldots, n$ in this case), is an $H-$matrix; then, most of the classical iterative methods for the solution of the problem at hand converge. In recent years the set of $H-$matrices has been extended to what is now known as the set of General $H-$matrices, and a partition of this set in three different classes has been made. The main objective of this work is to develop an algorithm that will determine the $H-$matrix character and will identify the class to which a given matrix $A \in \mathbb{C}^{n,n}$ belongs; in addition, some results on the classes of general $H-$matrices and a partition of the non-$H-$matrices set are presented.

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1 Introduction

The class of $H-$matrices (see, e.g., [4], [8], [13]) play a very important role in Numerical Analysis, in Optimization Theory and in other Applied Sciences. Suppose that $A \in \mathbb{C}^{n,n}$ is the coefficient of a linear system of algebraic equations. Then, A being an $H-$matrix constitutes a sufficient condition for a classical iterative method, like Jabobi, Gauss-Seidel, etc., to converge (see, e.g., [4], [13]). Also, given a Linear Complementarity Problem (LCP) whose matrix coefficient is a real $H-$matrix with a positive diagonal (H_+-matrix) guarantees that the LCP in question possesses a unique solution (see, e.g., [1], [4]).

For the identification of the $H-$matrix character of an $A \in \mathbb{C}^{n,n}$ in the Ostrowski's sense [10] (see, e.g., Varga [13]) many *direct* and *iterative* Algorithms have been developed. We mention those by Alanelli and Hadjidimos [2] and [3] and the references therein. However, in recent years this definition for $H-$matrices has been extended to encompass a wider set known as the set of *general $H-$matrices*. In a recent paper, [5], a partition of the general $H-$matrix set, $\mathcal{H}$, into three

*The present work is dedicated to Hans Schneider on the occasion of his 82nd birthday.

mutually exclusive classes was obtained: the *Invertible class*, $\mathcal{H}_I$, where all general H-matrices are nonsingular, the *Singular class*, $\mathcal{H}_S$, formed only by singular H-matrices, and the *Mixed class*, $\mathcal{H}_M$, in which singular and nonsingular $H-$matrices coexist. General $H-$matrices that are nonsingular have different properties when they belong to one of the classes $\mathcal{H}_I$ or $\mathcal{H}_M$. The same is true for general $H-$matrices that are singular and belong to the $\mathcal{H}_S$ or the $\mathcal{H}_M$ class. Consequently, some results referring to "nonsingular $H-$matrices", that is $H-$matrices that are nonsingular, and to "singular $H-$matrices", that is $H-$matrices that are singular must be revised.

The algorithm presented in [2], **AH** (Algorithm H), determines the $H-$matrix character of a given *irreducible* matrix A if it belongs to the invertible class $\mathcal{H}_I$, while **AH2** [3] covered the *reducible* case as well. These algorithms, based in the Ostrowski's definition for $H-$matrices, determine bounds for the spectral radius of the Jacobi iteration matrix associated with the comparison matrix of A. As will be seen, AH can also identify irreducible general $H-$matrices in the mixed class $\mathcal{H}_M$. General $H-$matrices in the singular class $\mathcal{H}_S$ can not be determined using the aforementioned algorithms since their Jacobi iteration matrix does not exist. Also, the general non-$H-$matrix character in the reducible case remains to be settled. To the best of our knowledge **no** Algorithm to identify a general $H-$matrix has been given so far and this is the main objective of the present work. However, in order to simplify the terminology and clarify the notation, from now on, the term *$H-$matrices* will refer to general $H-$matrices.

In this work we construct a new Algorithm, exploiting AH, which identifies the $H-$matrix character of $A \in \mathbb{C}^{n,n}$, that may be singular and/or reducible, makes the distinction among the three classes, as in [5], and the identification is made in a very efficient way.

To obtain the $H-$matrix character of a reducible matrix A, it suffices to study the character of the irreducible diagonal submatrices of its Frobenius normal form (**Fnf**) (see [5]). Our algorithm will use only the diagonal blocks of an Fnf and will not need to compute the Fnf itself. For this, some techniques are applied, combining irreducibility and $H-$matrix properties, avoiding at the same time unnecessary computations.

Apart from some initial and intermediate steps, our Algorithm consists of three main parts:

i) In the first part, it finds the irreducible/reducible character of a given matrix $A \in \mathbb{C}^{n,n}$.

ii) In the second part, which is skipped when A is irreducible, it finds a block permutation of the block diagonal, **bdFnf**, of an Fnf of A.

iii) In the third part, a slight modification of AH [2] is applied, if needed, to identify the $\mathcal{H}-$ or non-$\mathcal{H}-$matrix character and the class to which A belongs.

It should be pointed out that in case A is reducible this modification of AH is applied, if needed, to one or more irreducible diagonal blocks of order ≥ 2 of a bdFnf of A instead of to A itself. This reduces drastically the operations required for part (iii) and, in view of part (ii), our algorithm becomes more stable since fewer calculations are performed.

Note that to find an Fnf of a reducible A is **not** necessary when we are concerned with the localization of the eigenvalues of a reducible matrix by the *Geršgorin Circles*, by the *Brauer's Ovals of Cassini* or by the *Brualdi Lemniscates*, etc. (see Varga [15]) since only a bdFnf suffices. Using part (ii) unnecessary searches are avoided and a bdFnf is determined.

The outline of this work is as follows: In Section 2, basic notation and definitions are given and the required background material is presented. Also, some new results on classes of $H-$ and non-$H-$matrices are established. In Section 3, parts (i) and (ii) of our algorithm are theoretically developed. In Section 4, collecting all previous results and using AH, if needed, it is found out to which specific class of $H-$ or non-$H-$matrices a certain $A \in \mathbb{C}^{n,n}$ belongs. In Section 5, numerical examples are given in support of our theory while Section 6 concludes our work with some remarks.

2 Preliminaries and Basic Background Material

We begin with some notation, definitions and results, most of which can be found in [4] or [8]. Let $A \in \mathbb{C}^{n,n}$. For $n \in \mathbb{N}$, let $N := \{1, 2, 3, \cdots, n\}$. The symbol $|X|$, $X \in \mathbb{C}^{n,n}$, denotes the matrix whose elements are the *moduli* of the corresponding elements of X. The expression $A \geq 0$ ($A > 0$) denotes that A is a *nonnegative* (*positive*) matrix, i.e., $a_{ij} \geq 0$ ($a_{ij} > 0$, respectively), $i, j \in N$. The *Comparison Matrix* of $A \in \mathbb{C}^{n,n}$ is defined as $\mathcal{M}(A)$ with elements $m_{ii} = |a_{ii}|$ for all $i \in N$ and $m_{ij} = -|a_{ij}|$ for all $i \neq j \in N$. The set of *equimodular matrices* associated with A is the set $\Omega(A) := \{B \in \mathbb{C}^{n,n} : \mathcal{M}(B) = \mathcal{M}(A)\}$ [14]. If $D_A = \text{diag}(A)$ is invertible, the *Jacobi iteration matrix associated with* A is denoted by $J_A = I - D_A^{-1}A$. Further, in this case, the Jacobi iteration matrix associated with $\mathcal{M}(A)$ also exists and is the nonnegative matrix

$$J_{\mathcal{M}(A)} = \left|D_A^{-1}A\right| - I = |J_A|. \tag{2.1}$$

With these notations, if $\tau := \max_i\{a_{ii}\}$, the comparison matrix $\mathcal{M}(A)$ can be written as $\mathcal{M}(A) = \tau I - C$ where the matrix C is nonnegative ($C \geq 0$).

To the best of our knowledge, it was Schneider [11] who first extended the class of nonsingular $M-$matrices to the class of singular $M-$matrices which, in turn, led naturally to the class of general $M-$matrices (see [4]). So, based on all this we can give the definition for an $H-$matrix in the general sense (*general $H-$matrix*) as follows:

Definition 2.1. *A matrix $A \in \mathbb{C}^{n,n}$ is an $H-$matrix iff*

$$\mathcal{M}(A) = sI - B \quad \textit{with} \quad B \geq 0 \quad \textit{and} \quad s \geq \rho(B), \tag{2.2}$$

with $\rho(\cdot)$ denoting spectral radius. Particularly, if $A = \mathcal{M}(A)$, A is said to be an $M-$matrix.

Note that, in equation (2.2), $B \geq O \quad \Leftrightarrow \quad s \geq \tau = \max_i |a_{ii}|$.

In [5], a partition of the $H-$matrix set into three *mutually exclusive classes* was made as follows

$$\mathcal{H} = \mathcal{H}_I \cup \mathcal{H}_M \cup \mathcal{H}_S$$

and the main results related to our objective are summarized below.

(i) $\mathcal{H}_\mathbf{I}$: **Invertible class**. This class is characterized by the non-singularity of all matrices in the equimodular set. Among other interesting characterizations in [4], it is determined by $s > \rho(B)$ in (2.2), and, using the associated Jacobi iteration matrix we have:

$$A \in \mathcal{H}_I \quad \Leftrightarrow \quad D_A \text{ is nonsingular and } \rho(|J_A|) < 1. \tag{2.3}$$

Note that although this class contains only nonsingular $H-$matrices it does not contain all nonsingular ones. Nevertheless, it contains all nonsingular $M-$matrices.

(ii) $\mathcal{H}_\mathbf{M}$: **Mixed class**. This class is characterized with the observation that the equimodular set contains singular and nonsingular matrices (for $n > 1$ [1]). This class can also be characterized

[1]If $n = 1$ there exist only two classes: $[0] \in \mathcal{H}_S$ and $[x] \in \mathcal{H}_I$ for $x \neq 0$

(see Table 1 of [5]) by: D_A is nonsingular and $s = \rho(B)$ in (2.2). Moreover, $\mathcal{M}(A)$ is a singular $M-$matrix, and we will also use their Jacobi iteration matrix characterization:

$$A \in \mathcal{H}_M \quad \Leftrightarrow \quad D_A \text{ is nonsingular and } \rho(|J_A|) = 1. \tag{2.4}$$

In this class there exist some singular $H-$matrices (comparison matrices and others) but also there exist a larger number of nonsingular ones as we comment later.

(iii) $\mathcal{H}_\mathbf{S}$: **Singular class**. This class is characterized by the singularity of all matrices in the equimodular set and it is determined by: D_A is singular and $\mathcal{M}(A)$ is a singular $M-$matrix. Since the Jacobi iteration matrix J_A for these $H-$matrices does not exist, we will made a different characterization in Section 2.1 (Theorem 2.1).

Remark: In [6] the following result is established: if $A \in \mathcal{H}_M$ is irreducible

$B \in \Omega(A)$ *is singular* $\quad \Leftrightarrow \quad B = D_1\mathcal{M}(A)D_2,$ *where* D_1 *and* D_2 *are diagonal unitary matrices.*

So, in the equimodular set $\Omega(A)$ the subset of singular $H-$matrices consists only of matrices diagonally equivalent to $\mathcal{M}(A)$. Let us see why the number of nonsingular $H-$matrices in $\mathcal{H}_M$ is *much larger* than that of singular ones. Consider a totally dense singular $M-$matrix $A = \mathcal{M}(A) \in \mathcal{H}_M$. Taking all possible combinations of $\pm$ signs for each element of A, it is checked that there exist another $2^{n^2} - 1$ *real* matrices in $\Omega(A)$. However, by considering all possible products $D_1\mathcal{M}(A)D_2$, where D_1, D_2 are real diagonal unitary matrices, it is found that $\frac{1}{2}2^n \times 2^n = 2^{2n-1}$ of them are singular matrices. Taking the ratio $\frac{2^{2n-1}}{2^{n^2}} = \frac{1}{2^{(n-1)^2}}$ it is realized that, especially when the number n is large, the singular matrices constitute only a small percentage of the total number of real equimodular $H-$matrices of the $\mathcal{H}_M$ class.

From the exposed partition of the $H-$matrix set, we conclude a particular characterization of some non-$H-$matrices:

$$\rho(|J_A|) > 1 \quad \Leftrightarrow \quad A \in {}_n\mathcal{H}^{\emptyset}, \tag{2.5}$$

where ${}_n\mathcal{H}^{\emptyset}$ denotes the subset of the *non-$H-$matrix set*, ${}_n\mathcal{H}$, whose *main diagonal is nonsingular.* This set is different from the non-$H-$matrices that have *some null diagonal entry.* The latter subset is denoted by ${}_n\mathcal{H}^0$.

Then, considering the results of [2], we can conclude that Algorithm H, **AH**, can determine $H-$matrices in $\mathcal{H}_I$, irreducible matrices in $\mathcal{H}_M$ and non-$H-$matrices in ${}_n\mathcal{H}^{\emptyset}$. So, we need to distinguish $H-$matrices in $\mathcal{H}_S$ and non-$H-$matrices in the subset ${}_n\mathcal{H}^0$. To do this we summarize some results related to $H-$matrices and reducibility.

2.1 Reducible $H-$matrices

As it is known any $A \in \mathbb{C}^{n,n}$, $n \geq 2$, can be put, by a similarity permutation, in an upper triangular block form (*Frobenius normal form* (Fnf)), not necessarily unique, as follows:

$$\mathcal{F}(A) = PAP^T = \begin{bmatrix} F_{11} & F_{12} & F_{13} & \cdots & F_{1,q-1} & F_{1q} \\ 0 & F_{22} & F_{23} & \cdots & F_{2,q-1} & F_{2q} \\ 0 & 0 & F_{33} & \cdots & F_{3,q-1} & F_{3q} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & F_{q-1,q-1} & F_{q-1,q} \\ 0 & 0 & 0 & \cdots & 0 & F_{qq} \end{bmatrix}, \tag{2.6}$$

where $F_{ii} \in \mathbb{C}^{n_i,n_i}$ for $i \in Q := \{1, 2, \ldots, q\}$ such that $\sum_{i=1}^{q} n_i = n$ and where each F_{ii} is irreducible unless it is a 1×1 zero block. If $q = 1$ and $A \neq [0]$, A is *irreducible*, otherwise it is *reducible*.

We recall the following results from [5] on reducible $H-$matrices:

Let $A \in \mathbb{C}^{n,n}$ be a reducible matrix. Then, A is an $H-$matrix *iff* in the block form of the Fnf of A, $\mathcal{F}(A) = [F_{ij}]$ (2.6), each diagonal block F_{ii} is an $H-$matrix for all $i \in Q$. In addition, if $A \in \mathcal{H}$, $A \in \mathcal{H}_I \quad \Leftrightarrow \quad F_{ii} \in \mathcal{H}_I \quad \forall i \in Q$; $A \in \mathcal{H}_S \quad \Leftrightarrow \quad$ at least one $F_{ii} \in \mathcal{H}_S$; and $A \in \mathcal{H}_M \quad \Leftrightarrow \quad F_{ii} \notin \mathcal{H}_S \quad \forall i \in Q$ and at least one $F_{ii} \in \mathcal{H}_M$.

From [5, Theorem 3] we can readily determine a particular subset of non-$H-$matrices:

Lemma 2.1. *Let $A \in \mathbb{C}^{n,n}$ be irreducible such that D_A is singular. Then $A \in {}_n\mathcal{H}^0$.*

Consequently, a matrix $A \in \mathcal{H}_S$ is necessarily reducible. Moreover, null diagonal entries determine null 1×1 diagonal blocks in the Fnf of A and the remainder diagonal blocks are irreducible $H-$matrices:

Theorem 2.1. *Let $A \in \mathbb{C}^{n,n}$ and let $Z_D = \{i \ : \ a_{ii} = 0\}$. Then, $A \in \mathcal{H}_S$ iff $\operatorname{card}(Z_D) = s \geq 1$, A is reducible and the diagonal blocks F_{ii} of the Fnf of A (2.6) are in general of two types:*
(a) $F_{ii} = [0] = [a_{jj}]$ such that $j \in Z_D$, and this holds for precisely s diagonal blocks. If $s = n$, then $q = n$ and $\mathcal{F}(A)$ is an upper triangular matrix with null main diagonal.
(b) If $s < n$, for each $i \in Q \backslash Z_D$, $J_{F_{ii}}$ exists and $\rho\left(|J_{F_{ii}}|\right) \leq 1$.

Summarizing, we can characterize the non-$H-$matrices set ${}_n\mathcal{H}$:

$$A \in {}_n\mathcal{H} \Leftrightarrow \text{either the main diagonal of } F_{ii} \text{ is singular or } \rho\left(|J_{F_{ii}}|\right) > 1$$

for at least one irreducible diagonal block F_{ii} of the Fnf of A.

Nevertheless, since both conditions may hold in a matrix and some 1×1 diagonal blocks can also be null, we make a partition of ${}_n\mathcal{H}$ into three mutually exclusive classes:

Theorem 2.2. *If A is not an $H-$matrix, it belongs to one, and only one, of the following classes:*

$\mathbb{A} \in {}_{\ltimes}\mathcal{H}^{\nvDash} \Leftrightarrow D_A$ *is nonsingular and* $\rho(|J_A|) > 1$.

$\mathbb{A} \in {}_{\ltimes}\mathcal{H}_{\mathbb{S}}^{\nvdash} \Leftrightarrow D_A$ *is singular and A is reducible with each zero diagonal entry forming a 1×1 diagonal block of the Fnf of A, and, if F_{ii} is an irreducible diagonal block, the Jacobi iteration matrix associated with $\mathcal{M}(F_{ii})$, $J_{\mathcal{M}(F_{ii})} = |J_{F_{ii}}|$, exists and there exist(s) some irreducible diagonal block such that $\rho\left(|J_{F_{i_0 i_0}}|\right) > 1$. All matrices in this class are singular and belong to ${}_n\mathcal{H}^0$.*

$\mathbb{A} \in {}_{\ltimes}\mathcal{H}_{\mathbb{N}}^{\nvdash} \Leftrightarrow D_A$ *is singular with at least one null diagonal entry in an irreducible diagonal block F_{ii} of the Fnf of A (2.6) (If A is irreducible, $F_{ii} = A$). This is the complementary class of ${}_n\mathcal{H}_S^0$ in ${}_n\mathcal{H}^0$. The matrices in this class can be singular/reducible or not.*
Hence, we have the following partitions of the non-$H-$matrix set:

$$ {}_n\mathcal{H} = {}_n\mathcal{H}^{\emptyset} \cup {}_n\mathcal{H}^0 = {}_n\mathcal{H}^{\emptyset} \cup {}_n\mathcal{H}_S^0 \cup {}_n\mathcal{H}_N^0. $$

Proof: The proof follows by noting that ${}_n\mathcal{H}_S^0 \cap {}_n\mathcal{H}_N^0 = \emptyset$. $\square$

So, in order to distinguish the classes ${}_n\mathcal{H}_N^0$ and ${}_n\mathcal{H}_S^0$, we state and prove the following lemma.

Lemma 2.2. *Let $A \in \mathbb{C}^{n,n}$ be reducible with D_A singular. Then*

$$a_{ii} = 0 \text{ and } \sum_{k=1}^{n} |a_{ik} a_{ki}| > 0 \quad \Rightarrow \quad A \in {}_n\mathcal{H}_N^0. \tag{2.7}$$

Proof: $a_{ii}=0$ and $\sum_{k=1}^{n}|a_{ik}a_{ki}|>0 \Rightarrow a_{ik}a_{ki}\neq 0$ for some $k\neq i$. Then $a_{ii}=0$ is a diagonal entry of some irreducible diagonal block of the Fnf of A. By Lemma 2.1, A is not an $H-$matrix and, particularly, A belongs to the class ${}_n\mathcal{H}_N^0$ defined in Theorem 2.2. □

Condition (2.7) does not constitute a characterization of matrices in ${}_n\mathcal{H}_N^0$ as the following example shows.

Example 1.

$$A=\left[\begin{array}{c|ccc} 7 & 1 & 2 & 3\\ \hline 0 & 0 & 1 & 0\\ 0 & 0 & 2 & 1\\ 0 & 1 & 1 & 3\end{array}\right]\in {}_n\mathcal{H}_N^0$$

This matrix A is reducible and is written in Fnf. $F_{11}=[a_{11}]=[7]\in\mathcal{H}_I$ and $F_{22}\in {}_n\mathcal{H}_N^0$ by Lemma 2.1. Note that A and $F_{22}\in {}_n\mathcal{H}_N^0$ but $\sum_{k=1}^{n}|a_{2k}a_{k2}|=\sum_{k=2}^{n}|a_{2k}a_{k2}|=0$. In Section 3 we will give a complete characterization, Lemma 3.2, of 1×1 null diagonal blocks in the Fnf of a matrix A.

In the sequel we present some Examples where all possible cases of interest will be exhibited. For simplicity we consider matrices that coincide with their comparison counterparts and their entries are selected in a convenient way so that immediate conclusions can be drawn. Reducible matrices undergone a similarity permutation beforehand such that their Fnf is shown.

Example 2 ($H-$matrices: $A_1\in\mathcal{H}_I$, $A_2\in\mathcal{H}_M$, $A_3\in\mathcal{H}_S$).

$$A_1=\mathcal{M}(A_1)=\begin{bmatrix} 4 & -1 & -1 & -1\\ -1 & 4 & -1 & -1\\ -1 & -1 & 4 & -1\\ -1 & -1 & -1 & 4\end{bmatrix}=4I-\underbrace{\begin{bmatrix} 0 & 1 & 1 & 1\\ 1 & 0 & 1 & 1\\ 1 & 1 & 0 & 1\\ 1 & 1 & 1 & 0\end{bmatrix}}_{B_1}\ :\ s_1=4>\rho(B_1)=3$$

$$A_2=\mathcal{M}(A_2)=\begin{bmatrix} 3 & -1 & -1 & -1\\ -1 & 3 & -1 & -1\\ 0 & -2 & 3 & -1\\ 0 & -1 & -2 & 3\end{bmatrix}=3I-\underbrace{\begin{bmatrix} 0 & 1 & 1 & 1\\ 1 & 0 & 1 & 1\\ 0 & 2 & 0 & 1\\ 0 & 1 & 2 & 0\end{bmatrix}}_{B_2}\ :\ s_2=3=\rho(B_2)=3$$

$$A_3=\mathcal{M}(A_3)=\begin{bmatrix} 2 & -2 & -1 & -1\\ -2 & 2 & 0 & -1\\ 0 & 0 & 0 & -1\\ 0 & 0 & 0 & 5\end{bmatrix}=5I-\underbrace{\begin{bmatrix} 3 & 2 & 1 & 1\\ 2 & 3 & 0 & 1\\ 0 & 0 & 5 & 1\\ 0 & 0 & 0 & 0\end{bmatrix}}_{B_3}\ :\ s_3=5=\rho(B_3)=5$$

Note that A_1 is irreducible and nonsingular. The Jacobi iteration matrix associated with A_1 is $J_1=\frac{1}{4}B_1$ and $\rho(|J_1|)=\frac{3}{4}<1$. Then, all matrices in $\Omega(A_1)$ are nonsingular $H-$matrices.
A_2 is a singular $M-$matrix but its Jacobi iteration matrix is $J_2=\frac{1}{3}B_2$ and $\rho(|J_2|)=1$. The distinction between A_2 and A_3, despite that in both we have $s=\rho(B)$, is that A_3 is reducible, already in its Fnf, and has one 1×1 zero diagonal block. Note that in $\Omega(A_2)$ there exist some nonsingular matrices: $|A_2|$ is an example of such a nonsingular matrix. On the contrary, all matrices in $\Omega(A_3)$ are singular matrices. One notes that the bdFnf of its Fnf consists of the blocks $F_{11}=\begin{bmatrix} 2 & -2\\ -2 & 2\end{bmatrix}\in\mathcal{H}_M$, $F_{22}=[0]\in\mathcal{H}_S$ and $F_{33}=[5]\in\mathcal{H}_I$. Theorem 2.1 states that $A_3\in\mathcal{H}_S$.

Example 3 (Non-$H-$matrices: $A_4 \in {}_n\mathcal{H}^{\emptyset}$, $A_6 \in {}_n\mathcal{H}_S^0$, $A_5, A_7 \in {}_n\mathcal{H}_N^0$).

$$A_4 = \mathcal{M}(A_4) = \begin{bmatrix} 2 & -1 & -3 \\ -4 & 4 & -2 \\ 0 & -9 & 7 \end{bmatrix} = 7I - \underbrace{\begin{bmatrix} 5 & 1 & 3 \\ 4 & 3 & 2 \\ 0 & 9 & 0 \end{bmatrix}}_{B_4} \;:\; s_4 = 7 < \rho(B_4) = 9$$

$$A_5 = \mathcal{M}(A_5) = \begin{bmatrix} 4 & -1 & -1 \\ -2 & 0 & -2 \\ -1 & -1 & 4 \end{bmatrix} = 4I - \underbrace{\begin{bmatrix} 0 & 1 & 1 \\ 2 & 4 & 2 \\ 1 & 1 & 0 \end{bmatrix}}_{B_4} \;:\; s_4 = 4 < \rho(B_4) = 5$$

$$A_6 = \mathcal{M}(A_6) = \left[\begin{array}{c|c} [0] & -|X| \\ \hline 0 & A_4 \end{array}\right] = 7I - \underbrace{\left[\begin{array}{c|c} [7] & |X| \\ \hline 0 & B_4 \end{array}\right]}_{B_6} \;:\; s_6 = 7 < \rho(B_6) = 9 \quad \forall X \in \mathbb{C}^{1,3}$$

$$A_7 = \mathcal{M}(A_7) = \left[\begin{array}{c|c} [|a|] & -|X| \\ \hline 0 & A_5 \end{array}\right] = 4I - \underbrace{\left[\begin{array}{c|c} [4-|a|] & |X| \\ \hline 0 & B_5 \end{array}\right]}_{B_7} \;:\; s_7 = 4 < \rho(B_7) = 5 \quad \begin{array}{c} \forall X \in \mathbb{C}^{1,3} \\ |a| \le 4 \end{array}$$

Matrices of Example 3 are different in nature although for all of them there holds $s < \rho(B)$, then all are non-$H-$matrices. Firstly, the first two matrices are irreducible matrices and the other are not. Secondly, the particular difference between A_4 and A_5 matrices is the existence of a null diagonal entry in A_5. Then, the non-$H-$matrix character of A_4 may be concluded using its Jacobi iteration matrix, $\rho(|J_4|) > 1$, but the non-$H-$matrix character of A_5 follows from Lemma 2.1 since it is irreducible. Thirdly, the remainder matrices are reducible matrices and their diagonal matrices are singular. Then, to determine the non-$H-$matrix character, without computing $\rho(B)$, one needs to know their Fnf's. Fourthly, A_6 is in ${}_n\mathcal{H}_S^0$ since the only null diagonal entry is in a 1×1 block and we can compute the Jacobi iteration matrix of the other diagonal block, being again $\rho(|J_4|) > 1$. Finally, A_7 belong to ${}_n\mathcal{H}_N^0$ since one diagonal entry belongs to an irreducible diagonal block. Note that $a = 0$ in the matrix A_7 determines a null diagonal block in the Fnf of A_7 but this is not a contradiction.

The discussion so far reveals that in order to make the desired identification for a given $A \in \mathbb{C}^{n,n}$ one has to know whether A is **irreducible** or **reducible**. If it is irreducible, the identification is straightforward: either D_A is singular, $A \in {}_n\mathcal{H}_N^0$, or, otherwise, one can determine, using AH, if $\rho(|J_A|)$ is less than, equal to, or greater than 1 to conclude that A belongs to $\mathcal{H}_I$, $\mathcal{H}_M$ or ${}_n\mathcal{H}^{\emptyset}$, respectively. If A is reducible, one needs to know a bdFnf in order to determine to which class of $H-$ or non-$H-$matrices it belongs, applying, if needed, AH to one or more irreducible diagonal blocks of order ≥ 2 having in mind the results exposed.

3 Irreducible diagonal blocks

3.1 Is $A \in \mathbb{C}^{n,n}$ Irreducible or Reducible?

A statement that enables us to decide about the irreducibility of a given $A \in \mathbb{C}^{n,n}$ is the one below which is stated and proved in Lemma 2.2 of [13]. Specifically:

Lemma 3.1. *Let $A \in \mathbb{R}^{n,n}$ be irreducible with $A \geq 0$, then $(I + A)^{n-1} \; > \; O$.*

Based on the above lemma, we can easily prove the following theorem:

Theorem 3.1. *A matrix $A \in \mathbb{C}^{n,n}$ is irreducible iff $(I + |A|)^{n-1} > O$.*

Proof: If A is irreducible then so is $|A|$ ($\geq O$) and Lemma 3.1 applies. Conversely, let $(I + |A|)^{n-1} > O$ and A be reducible. Then, there exists a permutation matrix P such that $F = PAP^T = \begin{bmatrix} F_{11} & F_{12} \\ O & F_{22}, \end{bmatrix}$, where $F_{ii} \in \mathbb{C}^{n_i,n_i}$, $i = 1, 2$, with $n_1 + n_2 = n$. Forming $(I + |F|)^{n-1}$ we have

$$(I + |F|)^{n-1} = \begin{bmatrix} I_{n_1} + |F_{11}| & |F_{12}| \\ O & I_{n_2} + |F_{22}| \end{bmatrix}^{n-1} = \begin{bmatrix} (I_{n_1} + |F_{11}|)^{n-1} & |G_{12}| \\ O & (I_{n_2} + |F_{22}|)^{n-1} \end{bmatrix} \geq O.$$

Hence $(I+|A|)^{n-1} = (I+P^T\,|F|\,P)^{n-1} = P^T(I+|F|)^{n-1}P \geq O$, with strict inequality **not** holding, which contradicts our assumption. $\square$

Since we are only interested in the nonzero pattern of the powers of $I + |A|$, we may use 1's in the place of the nonzero elements of $I + |A|$. (*Note:* The last suggestion was made to the third of the authors by Professor Richard Varga [16].) So, using the notation spones$(\cdot)$ for the replacement of the nonzero elements of a matrix by 1's, we may write $C \equiv C(A) = \text{spones}(I+|A|) = \text{spones}(I + \text{spones}(A))$. Also, to avoid unnecessary calculations in forming C^k for all $k \in N\backslash\{n\}$, we form only C^2, C^4, C^8, $\cdots$, $C^{(2^l)}$, with $l = \lceil \frac{\log(n-1)}{\log 2} \rceil$. For the same reason as before, we replace C^{2^k} with $\text{spones}\left(C^{2^k}\right)$ for each k. We stop forming $C^{(2^m)}$, $1 \leq m \leq l$, as soon as the number of nonzero elements (**nnz**) of any $C^{(2^m)}$ is n^2, in which case A is irreducible, or when a possible equality $\text{nnz}(C^{(2^{m-1})}) = \text{nnz}(C^{(2^m)}) < n^2$ occurs, in which case A is reducible. That is, from Theorem 3.1 we conclude the following result:

Corollary 3.1. *Let $A \in \mathbb{C}^{n,n}$, $l = \lceil \frac{\log(n-1)}{\log 2} \rceil$, $C_0 = \text{spones}\,(I + |A|)$ and $C_k = \text{spones}\left(C_{k-1}^2\right)$, $k = 1, 2, \ldots l$. Then, A is reducible iff either $\text{nnz}(C_k) = \text{nnz}(C_{k-1}) < n^2$ for some $k < l$ or $\text{nnz}(C_l) < n^2$. In other words, A is irreducible if and only if $\text{nnz}(C_k) = n^2$ for some $k \leq l$.*

Proof: If $\text{nnz}(C_k) = \text{nnz}(C_{k-1}) < n^2$, then $C_l = C_k = C_{k-1}$. Hence $\text{nnz}(C_l) < n^2$ and the proof follows. $\square$

Example 4. *For the matrix A below we form the matrices C_k as in Corollary 3.1 concluding that A is a reducible matrix.*

$$A = \begin{bmatrix} 0.8 & 0 & 0 & 0 & -1.2 & 0 \\ 0 & 0.9 & 0 & 0.6 & 0 & 0 \\ 0 & 0 & 0.7 & 0 & 0 & 0 \\ 0 & 0 & 0.1 & 0.5 & 0 & -0.1 \\ 0.3 & 0 & -1.0 & 0 & 0.6 & 0 \\ 0 & 1.1 & 0 & 0 & -0.7 & 1.0 \end{bmatrix} \quad \Rightarrow \quad C_0 = \begin{bmatrix} 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1 & 1 \end{bmatrix}.$$

Since $n = 6$, $l = \lceil \frac{\log(n-1)}{\log 2} \rceil = \lceil 2.3219 \rceil = 3$, we should form C_1, C_2 and C_3. Since $\text{nnz}(C_3) = \text{nnz}(C_2) = 25 < 6^2 = 36$ is obtained, A is reducible.

Then, if A is a reducible matrix, using the notation of Corollary 3.1, the last matrix computed is the densest between the powers of C_0 and coincides with C_l. This matrix will be used in Section 3.2. The matrix C_l obtained in Example 4 is

$$C_l = C_3 = \text{spones}(C_2^2) = C_2 = \begin{bmatrix} 1 & 0 & 1 & 0 & 1 & 0 \\ 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 0 & 1 & 0 \\ 1 & 1 & 1 & 1 & 1 & 1 \end{bmatrix}. \tag{3.1}$$

Example 5. *Let*

$$A = \begin{bmatrix} 0.1 & 0 & 0 & 0 & 0 & 0 & 0 & -0.2 & 0 \\ 1 & 9 & 8 & 5 & 4 & 3 & 6 & 2 & 7 \\ 0.7 & 0 & -0.8 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -0.1 & 0 & 0 & 0.2 & 0 & 0 \\ 0 & 0 & 0 & 0.8 & -0.7 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -0.6 & 0.5 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0.3 & 0 & -0.4 \\ 0 & 0 & 0 & 0 & 0 & 0.4 & 0 & -0.3 & 0 \\ 0 & 0 & 0.6 & 0 & 0 & 0 & 0 & 0 & -0.5 \end{bmatrix} \in \mathbb{R}^{9,9}.$$

Computing the matrices C_k as in Example 4 for $k = 0, 1, 2, 3$ we obtain that A is reducible since $\text{nnz}(C_3) = 73 < l^2 = 81$. *In this example* $\text{nnz}(C_2) = 49 < \text{nnz}(C_3)$.

Then, the first part of the new algorithm (*IRR-algorithm*), whose purpose is to determine the irreducible character of a square complex matrix, is the following:

Algorithm 3.1 (IRR)**.**

1. $Z_D = \{i \ : \ a_{ii} = 0\}$, $NH = 0$, $IRR = 0$, $C = I + \text{spones}(A)$
2. *FOR* $i \in Z_D$ *: IF* $\sum_k |a_{ik}a_{ki}| \neq 0$ *: "*$A \in {}_n\mathcal{H}_N^0$*",* $NH = 1$ ***END OF TOTAL PROCESS***
3. *IF* $\text{nnz}(C) = n^2$ *: "A is irreducible",* $IRR = 1$*,* ***END***
4. $l = \lceil \frac{\log(n-1)}{\log 2} \rceil$, $\quad k = 0$
5. *WHILE* $NH = 0$ *and* $IRR = 0$ *:*
6. $\quad p = \text{nnz}(C)$, $B = C^2$, $q = \text{nnz}(B)$, $k = k + 1$
7. $\quad$ *FOR* $i \in Z_D$ *: IF* $b_{ii} > 1$ *: "*$A \in {}_n\mathcal{H}_N^0$*",* $NH = 1$ ***END OF TOTAL PROCESS***
8. $\quad$ *IF* $q = n^2$ *AND* $Z_D \neq \emptyset$ *: "*$A \in {}_n\mathcal{H}_N^0$*",* $NH = 1$ ***END OF TOTAL PROCESS***
9. $\quad$ *ELSE IF* $q = n^2$ *AND* $Z_D = \emptyset$ *"A is irreducible",* $IRR = 1$*,* ***END***
10. $\quad$ *ELSE IF* $p = q$ $(< n^2)$ *OR* $k = l$ *: "A is reducible",* $IRR = -1$*, SAVE matrix* C*,* ***END***
11. $\quad$ *ELSE :* $C = \text{spones}(B)$ $\quad$ *(RETURN TO STEP 5)*

According to Corollary 3.1 the natural end of the algorithm is $IRR = 1$ (irreducible) or $IRR = -1$ (reducible). Nevertheless, since our main objective is to determine the $H-$matrix character, the test of some conditions for null diagonal entries is added: In Step 2 following Lemma 2.2, in Step 8 Lemma 2.1, and in Step 7 following the lemma below.

Lemma 3.2. *Let $A \in \mathbb{C}^{n,n}$, $Z_D = \{i \ : \ a_{ii} = 0\} \neq \emptyset$ and $G = I + \text{spones}(A)$. Then, the following two conditions are equivalent:*
1. $[a_{ii}] = [0]$, for $i \in Z_D$, is a diagonal block of the Fnf of A.
2. The x_{ii} diagonal entry of G^k satisfies $x_{ii} = 1$ for $k = 1, 2, 3, \ldots, n$.

Proof: The Fnf's of A and of G have the same structure, that is, they have the same block sizes and the same index subsets that determine diagonal blocks. The same holds true for the Fnf's of G^k, $k = 1, 2, \ldots$, since $G \geq 0$ (no new zero entry can appear in the successive powers of G).
$1 \Rightarrow 2$: Note that, $a_{ii} = 0 \quad \Leftrightarrow \quad g_{ii} = 1$. Moreover, if $k = 2$, the diagonal entry x_{ii} of G^2 is

$$x_{ii} = \sum_k g_{ik} g_{ki} = 1 + \sum_{k \neq i} g_{ik} g_{ki} = 1$$

because $[g_{ii}]$ is a 1×1 diagonal block. Computing the successive powers of G we conclude that $[a_{ii}] = [0]$ is a diagonal block of the Fnf of A and hence $[x_{ii}] = [1]$ is a diagonal block of the Fnf of G^k for all k.

$2 \Rightarrow 1$: Suppose that $i = 1$ and that an irreducible diagonal block of the Fnf of G is the submatrix $M = [g_{ij}]$ for $i, j = 1, 2, \ldots, n_1$ such that $1 < n_1 \leq n$. By Theorem 3.1, $M^{n_1 - 1} = [m_{ij}] > 0$, and $m_{ij} \geq 1$. Then the x_{11} entry of G^{n_1} is $x_{11} \geq \sum_{j=1}^{n_1} m_{1j} m_{j1} \geq n_1 > 1$ which is a contradiction. □

We show the above results in the following example.

Example 6. *Let* $A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 2 & 1 \\ 1 & 1 & 3 \end{bmatrix} \in {}_n\mathcal{H}_N^0$ *the diagonal block F_{22} of the matrix of Example 1.*

Applying IRR-algorithm we have for $k = 1$, $p = \text{nnz}(C) = \text{nnz}(\text{spones}(I + |A|)) = 6 < n^2 = 9$. For $k = 2$ we compute

$$B = C^2 = \begin{bmatrix} 1 & 2 & 1 \\ 1 & 2 & 2 \\ 2 & 3 & 2 \end{bmatrix}$$

such that $b_{11} = 1$ and $\text{nnz}(B) = 9 = n^2$. Then A is an irreducible matrix but $A \in {}_n\mathcal{H}_N^0$ since $Z_D \neq \emptyset$.
However, if we compute $G = C^3$, the first diagonal entry of G is $x_{11} = 2 > 1$ as Lemma 3.2 predicts.
Applying IRR-algorithm to the matrix A of Example 1, we have, for $k = 1$,

$$C_1 = C_0^2 = \begin{bmatrix} 1 & 3 & 4 & 3 \\ 0 & 1 & 2 & 1 \\ 0 & 1 & 2 & 2 \\ 0 & 2 & 3 & 2 \end{bmatrix}$$

such that $\text{nnz}(B) = 13 < n^2$ and $b_{22} = 1$. Then, we compute $C_2 = \text{spones}(C_1^2)$, such that $\text{nnz}(C_2) = 13 = \text{nnz}(C_1)$, then A is reducible, but now $b_{22} = 3 > 1$ and then $A \in {}_n\mathcal{H}_N^0$ since $b_{22} > 1$ $\Rightarrow a_{22}$ does not determine a 1×1 diagonal block (Lemma 3.2).

Summarizing, IRR-Algorithm concludes as follows:
1. If $Z_D = \emptyset$ ($\Leftrightarrow D_A$ is nonsingular), then
either $IRR = 1$, hence A is irreducible and the matrices B and C are not used any further,

or $IRR = -1$, hence A is reducible and the final matrix C computed coincides with $C_l = \text{spones}(I + |A|)^{n-1}$ and it is saved for further use.

2. If $Z_D \neq \emptyset$ ($\Leftrightarrow D_A$ is singular), then
   either $NH = 1$, then A has some null diagonal entry in an irreducible principal submatrix (A being irreducible is included) and is **not** an $H-$matrix. Particularly, $A \in {}_n\mathcal{H}_N^0$. In such a case the total process ends with the identification of A being completed.
   or $NH = 0$ and $IRR = -1$, then A is reducible and the final matrix C is also saved. Moreover, all null diagonal entries are classified as 1×1 null diagonal blocks of the Fnf of A, and their indices are saved in the set Z_D.

Note that, applying IRR-algorithm, we could obtain that $A \in {}_n\mathcal{H}_N^0$; otherwise, we can remove the possible indices in Z_D and we can apply AH to the matrix with the remainder indices. Nevertheless, since AH can present problems with matrices belonging to $\mathcal{H}_M$, due to round-off errors, we will determine in an easy way the bdFnf of A using the saved matrix C in order to apply AH to the irreducible diagonal blocks of the Fnf of A.

3.2 Determination of the bdFnf of $A \in \mathbb{C}^{n,n}$

Theorem 3.2. *Let $A \in \mathbb{C}^{n,n}$ be reducible, $\mathcal{F}(A)$ the Fnf of A of (2.6) and $E = B + B^T$, where $B = \text{spones}\left((I + |\mathcal{F}(A)|)^{n-1}\right)$. Then, the matrix E, partitioned as in (2.6), will have diagonal blocks*

$$E_{ii} \;=\; 2\boldsymbol{e}_{n_i}\boldsymbol{e}_{n_i}^T \in \mathbb{R}^{n_i,n_i}, \quad \textit{where} \quad \boldsymbol{e}_{n_i} = [1\,1\,\cdots\,1]^T \in \mathbb{R}^{n_i},\ \forall i \in Q,$$

while its off-diagonal blocks E_{ij}, $\forall i \neq j \in Q$ will have as entries 1's and 0's.

Proof: The diagonal blocks $I_{n_i} + |\mathcal{F}(A)|_{n_i,n_i}$, $\forall i \in Q$, of $I + |\mathcal{F}(A)|$, are irreducible and nonnegative. So, by Lemma 3.1, we will have that $B_{ii} = \left((I + |\mathcal{F}(A)|)^{n-1}\right)_{n_i,n_i} > O$. Hence $E_{ii} = 2\mathbf{e}_{n_i}\mathbf{e}_{n_i}^T$, $\forall i \in Q$. The blocks B_{ij} in the block upper triangular part of B will satisfy $B_{ij} \geq O_{n_i,n_j}$ for $i \in Q\backslash\{q\}$, $j = i+1, \ldots, q$, while those of its block lower triangular part will be $B_{ij} = O_{n_i,n_j}$ $\forall i \in Q\backslash\{1\}$, $j = 1, \ldots, i-1$. Hence, the conclusions of the theorem readily follow. □

Obviously, if A is irreducible it is already in its Fnf. In case A is reducible to find a block permutation of the block diagonal of an Fnf of A we use some ideas from the *compact profile technique* in Kincaid et al. [9] and the *extended compact profile technique* in Hadjidimos [7]. Suppose A is a reducible matrix and C is the matrix obtained and saved by IRR-algorithm: $C = C_l = \text{spones}\left((I + |A|)^{n-1}\right)$. Let $\mathcal{F}(A)$ and E be the matrices as in Theorem 3.2 and let P the permutation such that $A = P\mathcal{F}(A)P^T$. If we form the matrix $R = C + C^T$, then $R = PEP^T$. Since the entries of the diagonal blocks of E are 2's and the remainder entries are 1's and 0's, the entries $r_{ij} = 2$ can be used to determine the indices of the respective diagonal blocks of $\mathcal{F}(A)$ and to obtain the bdFnf of A. To make clear how these ideas work in our case we give an example.

Example 7. *Let A be the reducible matrix of Example 4. One of its Fnf's is*

$$\mathcal{F}(A) \;=\; \left[\begin{array}{rrr|rr|r} 1.0 & 0 & 1.1 & -0.7 & 0 & 0 \\ -0.1 & 0.5 & 0 & 0 & 0 & 0.1 \\ 0 & 0.6 & 0.9 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0.6 & 0.3 & -1.0 \\ 0 & 0 & 0 & -1.2 & 0.8 & 0 \\ \hline 0 & 0 & 0 & 0 & 0 & 0.7 \end{array}\right].$$

(The similarity permutation needed to obtain $\mathcal{F}(A)$ is $\begin{pmatrix} 6 & 4 & 2 & 5 & 1 & 3 \end{pmatrix}$).

¿From (3.1) we get

$$R = C_3 + C_3^T \;=\; \begin{bmatrix} 2 & 1 & 1 & 1 & 2 & 1 \\ 1 & 2 & 1 & 2 & 1 & 2 \\ 1 & 1 & 2 & 1 & 1 & 1 \\ 1 & 2 & 1 & 2 & 1 & 2 \\ 2 & 1 & 1 & 1 & 2 & 1 \\ 1 & 2 & 1 & 2 & 1 & 2 \end{bmatrix}. \tag{3.2}$$

Then, indices 1 and 5 determine a diagonal block of order 2; indices 2, 4 and 6 determine a diagonal block of order 3; and index 3 determine a 1×1 diagonal block. This technique is described next and we show the results using the matrices of Examples 4 and 7.

We form the two vectors:

a) perm: a vector of size n which on input has $\text{perm}_i = i \;, \forall i \in N$. On exit it will contain a permutation of $(1, 2, \ldots, n)$ that will indicate the similarity permutation required to obtain a block permutation of an Fnf of A.

b) orbs: a vector of size $q \leq n$. On exit, orbs_i will be the order of the ith block of the bdFnf such that $\sum_{i=1}^{q} \text{orbs}_i = n$.

We begin by setting $\text{perm}_1 = 1$, and find the row/column indices of entries such that R_{i1}, (and R_{1i})$= 2$, $i \in N$. If I_1 is this index set and n_1 is its cardinality, i.e., the number of 2's found, we set $\text{orbs}_1 = n_1$ and interchange the contents of perm_i, for $i \in I_1$, with those of perm in the n_1 first positions. (If $n_1 = 1$, interchanges in perm are avoided.) That is, orbs_1 is the order of the first diagonal block and the n_1 first values of perm are the corresponding row/column indices having a value 2 in the first row/column of matrix R. In our example, only $R_{11} = R_{51} = 2$. So, $I_1 = \{1, 5\}$, $n_1 = 2$ and we need to interchange second and fifth positions. Then $\text{perm} = (1, 5, 3, 4, 2, 6)$ and $\text{orbs}_1 = 2$.

Next, we consider $j = \text{perm}_{n_1+1}$ ($j = \text{perm}_3 = 3$ in our example) and $I_2 = \{i \;:\; R_{ij} = R_{ji} = 2\}$. If $n_2 = \text{card}(I_2)$, we set $\text{orbs}_2 = n_2$ and interchange the contents of perm_i for $i \in I_2$ with those of perm in the positions $n_1 + 1$ until $n_1 + n_2$. In our example, $I_2 = \{3\}$, $n_2 = 1$, $\text{orbs}_2 = 1$ and, then, $\text{perm} = (1, 5, 3, 4, 2, 6)$ remains unaltered.

Continuing, we consider $j = n_1 + n_2 + 1$ and we obtain the respective I_3 and n_3. Particularly, it can be found that $n_3 = 3$ in our example and so $n_1 + n_2 + n_3 = 6 = n$. So, $\text{orbs}_3 = 3$, $\text{perm} = (1, 5, 3, 4, 2, 6)$ remains unchanged and the bdFnf can be obtained. On exit the vectors perm and orbs appear as in Table 1.

The bdFnf suggested by Table 1 is

perm	1	5	3	4	2	6
orbs	2		1	3		

Table 1: Final vectors perm and orbs for matrix A of Example 4

$$D = \left[\begin{array}{cc|c|ccc} 0.8 & -1.2 & 0 & 0 & 0 & 0 \\ 0.3 & 0.6 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0.7 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0.5 & 0 & -0.1 \\ 0 & 0 & 0 & 0.6 & 0.9 & 0 \\ 0 & 0 & 0 & 0 & 1.1 & 1.0 \end{array}\right]. \tag{3.3}$$

Note that the diagonal blocks numbered 1, 2, 3 in $\mathcal{F}(A)$ in Example 7 appear now as blocks 3, 1, 2 in D with inner similarity permutations in two of the blocks. Then the block diagonal matrix D in (3.3) is a bdFnf of A.

In addition, we can construct the bdFnf according to orbs where the diagonal blocks are taken in a non decreasing order. However, we do not need this particular bdFnf: we can analyze the $H-$matrix character of each diagonal block, in the third part of our algorithm, taking them following the orders indicated by orbs.

In conclusion, using IRR-algorithm, if $NH = 0$ and $IRR = -1$ is obtained, the densest matrix $C = \text{spones}((I + |A|)^{n-1})$ is saved. Next, the following algorithm (BD), computes $R = C + C^T$ and determines the vectors perm and orbs. Some temporary variables are used: "bn" for "block number"; "bo" for "block order" for "lpp" by "last permuted position". Moreover, only with the purpose of simplifying notation in the next expressions, the row indices defining a diagonal block are also determined and denoted by I_k. The first line defines the function $w = \text{PERMUTE}(i, j, v)$ which interchanges the positions v_i and v_j and the rest of entries remain unaltered: $w_k = v_k$ for $k \neq i, j$; $w_i = v_j$ and $w_j = v_i$.

Algorithm 3.2 (BD).

$\star$ *Declare function PERMUTE* : $v =$*PERMUTE*(i, j, v) *such that* $k = v(j)$, $v(j) = v(i)$, $v(i) = k$

1. *IF IRR*$= -1$ *and NH*$= 0$: $R = C + C^T$, perm $= (1, 2, \ldots, n)$, $s = \text{card}(Z_D)$, $lpp = s + 1$, $bn = s$
2. *FOR* $j = 1$ *TO* s : *Let* $i_j \in Z_D$, perm $=$*PERMUTE*(j, i_j, perm), $\text{orbs}_j = 1$, $I_j = (i_j)$
3. *WHILE* $lpp < n$
   $j = \text{perm}(lpp)$, $bn = bn + 1$, $bo = 1$
   FOR $i = 1$ *TO* n
   IF $j \neq i$ *and* $r_{ji} = 2$: perm $=$*PERMUTE*$(lpp + 1, i, \text{perm})$, $lpp = lpp + 1$, $bo = bo + 1$
   $\text{orbs}_{bn} = bo$, $I_{bn} = \text{perm}(j : lpp) = \{\text{perm}_j, \text{perm}_{j+1}, \ldots, \text{perm}_{lpp}\}$
4. $q = bn$ (*q is the number of diagonal blocks and it is the size of vector* orbs)

By saving the vectors perm and orbs (or the respective index set I_i, $\forall\, i \in Q$), we obtain a block permutation of the block diagonal of the Fnf of A shown in (2.6). Specifically, the following bdFnf:

$$D = [F_{ii}], \text{ where } F_{ii} \in \mathbb{C}^{n_i \times n_i}, \forall\, i \in Q. \tag{3.4}$$

That is, D is a block diagonal matrix such that $F_{ii} = [a_{jk}]$, where $j, k \in I_i$ and I_i contains $n_i = \text{orbs}_i$ indices, particularly the indices of perm in the positions $\sum_{m=1}^{i-1} n_m + 1$ to $\sum_{m=1}^{i} n_m$, $\forall i \in Q\backslash\{1\}$. In addition, $F_{ii} = [0]$ for $i = \text{perm}_j$, $j = 1, 2, \ldots, s$, that is, $n_i = \text{orbs}_i = 1$ for $i = 1, 2, \ldots, s$.

4 Determination of the $H-$matrix character of $A \in \mathbb{C}^{n,n}$

As it was mentioned, a slight modification of AH is to be applied to one or more irreducible diagonal blocks of the Fnf of A *iff* **no** immediate conclusion, as to which class of $H-$ or non-$H-$matrices the given matrix $A \in \mathbb{C}^{n,n}$ belongs, is obtained before. Without appealing to AH a conclusion can be drawn in the following two cases:
a) If the Fnf of A is triangular, in which case $A \in \mathcal{H}_I$ or $A \in \mathcal{H}_S$ if D_A is nonsingular or singular respectively, and
b) If A has an irreducible principal submatrix with a zero diagonal element, in which case $A \in {}_n\mathcal{H}_N^0$.

The reader is reminded that AH can be applied only to irreducible matrices with **no** zeros in their diagonal, and these irreducible diagonal blocks have been obtained by the BD-algorithm. So, the Modification suggested of AH to be applied to irreducible diagonal blocks without null diagonal entries is the following. The input matrix is a diagonal block $F = F_{ii} \in \mathbb{C}^{n_i \times n_i}$, with $n_i > 1$, and the output of this modification is only the parameter r:

Algorithm 4.1 (ModAH)**.**
1. $r = -1$, $J = \left|D_F^{-1}F\right| - I$ *(*$J = |J_F|$ *is the Jacobi iteration matrix associated with* $\mathcal{M}(F)$*)*
2. *maxiter* $= 1000$, $k = 1$
3. *WHILE* $r < 0$
4. *FOR* $i = 1$ *to* n_i *:* $S_i = \sum_j b_{ij}$
5. $m = \min S_i$, $M = \max S_i$
6. *IF* $m > 1$ *:* $r = m > 1$ *(*$F \in {}_n\mathcal{H}^{\emptyset}$*),* ***END***
7. *ELSE IF* $M < 1$ *:* $r = M < 1$ *(*$F \in \mathcal{H}_I$*),* ***END***
8. *ELSE IF* $m = M$ *(* $= 1$ *) :* $r = 1$ *(*$F \in \mathcal{H}_M$*),* ***END***
9. *ELSE :* $D = \text{diag}(1 + S_i)/(1 + M)$, $J = D^{-1}JD$, $k = k + 1$
10. *IF* $k >$ *maxiter :* ***STOP*** *(and send message)*
11. *ELSE (RETURN TO STEP 3)*

Then, the third part of the new algorithm collects the results of ModAH applied to the q diagonal blocks of the bdFnf of A (3.4) obtained by the BD-algorithm. Note that each 1×1 diagonal block is an $H-$matrix and $s = \text{card}(Z_D)$ of them are in $\mathcal{H}_S$. That is, Part 3 applies ModAH-algorithm to each diagonal block F_{ii} of order $n_i > 1$ determined by the indices in $I_i = \{\text{perm}_j \ : \ j = q+1, q+2, \ldots, q+n_i\}$ for $q = n_1 + n_2 + \cdots + n_{i-1}$. In case the output parameter r is greater than 1 for some F_{ii}, the original matrix A is not an $H-$matrix and the complete process terminates. In this case, we set NH$= 2$ and $A \in {}_n\mathcal{H}^{\emptyset}$ or $A \in {}_n\mathcal{H}_S^0$ if $s = 0$ or $s > 0$, respectively. If $r = 1$ for some F_{ii} we define a new variable MH$= 1$ (for Mixed $H-$matrix), in order to determine the class of A at the end of the process, while if $r < 1$ for all r_is MH$= 0$. Recall that from the first part, the IRR-algorithm determines all cases where $A \in {}_n\mathcal{H}_N^0$ setting NH$= 1$. Then, if NH remains null, the matrix A is an $H-$matrix, and belongs to $\mathcal{H}_S$ if $s > 0$, or to $\mathcal{H}_M$ if MH$= 1$ and $s = 0$, or $A \in \mathcal{H}_I$ if $s = 0$ and MH$= 0$.

So, the complete algorithm to determine the irreducible/reducible character as well as the $H-$/non-$H-$matrix character and the class of a general matrix $A \in \mathbb{C}^{n,n}$ is the given below. We call it Algorithm Based on General $H-$matrices, or simply **ABGH**, and it gives rise to an extension and modification of AH.

Algorithm 4.2 (ABGH)**.**

- *INPUT :* $A \in \mathbb{C}^{n,n}$

  1. *Apply the IRR-algorithm to A. The OUTPUT variables are : the index set Z_D of null diagonal entries, $s = \mathrm{card}(Z_D) \in [0, n]$, the temporary variable NH (initial value NH$= 0$), and the main variable IRR (initial value IRR$= 0$). Particular final results:*
     a) $A \in {}_n\mathcal{H}_N^0 \quad \Leftrightarrow \quad$ NH$= 1$.
     b) A is irreducible $\quad \Leftrightarrow \quad$ IRR$= 1$. A is reducible $\quad \Leftrightarrow \quad$ IRR$= -1$. In the latter case, the densest matrix C is included in the output.
  2. *If NH$= 0$ and $IRR = -1$: apply the BD-algorithm using as input C and Z_D. The output consists of the vectors* perm *and* orbs *(the number of diagonal blocks in the bdFnf is q; specifically, q is the size of vector* orbs*) and the respective index sets I_i, $i \in Q$, that determine the bdFnf: $F_{ii} = [a_{jk}]$, $j, k \in I_i$. (The sets I_i can also be determined from* perm *and* orbs.*)*
     If NH$= 0$ and $IRR = 1$: set $q = 1$, perm *$= (1, 2, \ldots, n)$, I_1 $=N$ and* orbs *$= n$. ($F_{11} = A$)*
  3. *If NH$= 0$, Part 3 determines the character of the successive diagonal blocks F_{ii} following the rule: if* orbs$_i$ *> 1, apply ModAH to F_{ii} and obtain the value of the parameter $r_i = r$*
     - *If $r_i > 1$: NH$= 2$, **END OF PART 3***
     - *If $r_i = 1$: MH$= 1$*
     - *If $r_i < 1$ for all i : MH$= 0$*

- *OUTPUT : Using the values of NH, MH and s, conclude to which class of $H-$matrix/non-$H-$matrix A belongs:*

  *if NH$= 0$: "A **is** an $H-$matrix" and*

  $A \in \mathcal{H}_I$ *iff* $s = 0$ *and* $MH = 0$; $A \in \mathcal{H}_M$ *iff* $s = 0$ *and* MH$= 1$; $A \in \mathcal{H}_S$ *iff* $s > 0$.

  *if NH> 0 : "A **is not** an $H-$matrix" and*

  $A \in {}_n\mathcal{H}_N^0$ *iff* NH$= 1$; $A \in {}_n\mathcal{H}^{\emptyset}$ *iff* NH$= 2$ *and* $s = 0$; $A \in {}_n\mathcal{H}_S^0$ *iff* NH$= 2$ *and* $s > 0$.

  Moreover, the conclusion on reducible/irreducible character of A (IRR$= -1/1$ respectively) is obtained unless $A \in {}_n\mathcal{H}_N^0$.

Based on the extended analysis of examples, on the notation and the theoretical analysis so far, as well as on what has been explained in relation with the Algorithms, we present the main theorem whose proof is given in an algorithmic way. It is simply noted that the diagonal blocks of the bdFnf at hand are denoted by B_{ii}, $i \in Q$. These may be those of $\mathcal{F}(A)$ in (2.6), $B_{ii} = F_{ii}$, or of a block permutation of $\mathcal{F}(A)$, $B_{ii} = F_{jj}$, and/or with inner permutations of them, $B_{ii} = P_j F_{jj} P_j^T$, or even $B_{ii} = A$ if A is irreducible.

Theorem 4.1. *Let $A \in \mathbb{C}^{n,n}$ be the input of ABGH. If A is an $H-$matrix, the Output of ABGH is correct and determines the particular class, $\mathcal{H}_I$, $\mathcal{H}_M$ or $\mathcal{H}_S$ to which it belongs. On the other* *hand, the Output of ABGH is also correct since it determines that A is not an $H-$matrix and, in addition, it determines to which class of non-$H-$matrices A belongs, that is ${}_n\mathcal{H}^{\emptyset}$, ${}_n\mathcal{H}_S^0$ or ${}_n\mathcal{H}_N^0$. Unless $A \in {}_n\mathcal{H}_N^0$, the reducible/irreducible character of A is also included in the output of the ABGH.*

Proof: If $A \in {}_n\mathcal{H}_N^0$, Part 1 concludes that NH$=1$, by virtue of Lemmas 2.2 and 3.2 and, on exit, the conclusion is shown; no more computations are required for this class of matrices.

Otherwise, NH remains null. If A is irreducible, Part 1 concludes that IRR$=1$ ($\text{nnz}(C^k) = n^2$ for $C = \text{spones}(I + |A|)$ and some k : Theorem 3.1). Otherwise, IRR$=-1$ and A is reducible ($\text{nnz}(C^l) < n^2$ for $l = \lceil \frac{\log(n-1)}{\log 2} \rceil$: Corollary 3.1).

For the reducible matrices only (IRR$=-1$), Part 2 determines a bdFnf of A, $D = [B_{ii}]$, $i = 1, 2, \ldots, q$, $q \in [2, n]$, such that, if the number of null 1×1 diagonal blocks of A is $SH = s$, the first s diagonal blocks are 1×1 null matrices. If $r = q - s$, r diagonal blocks are **irreducible** matrices and their main diagonals are nonsingular (Lemma 2.1). For irreducible matrices (IRR$=1$ and NH$=0$) the only irreducible diagonal block is A itself and its main diagonal is nonsingular. Then $D = A = B_{11}$, $q = r = 1$ and $s = 0$.

Then, Part 3 determines the $H-$matrix character and class based on the following notes:

1. A 1×1 matrix is an $H-$matrix. Then, A has $SH = s$ diagonal blocks in $\mathcal{H}_S$ and $IH = r_1$ diagonal blocks in $\mathcal{H}_I$ (being r_1 the number of 1×1 invertible diagonal blocks). Note that, in the particular case where $q = n = s + r_1$, the Fnf of A is a triangular matrix and then $A \in \mathcal{H}_I$ if $SH = 0$, or $A \in \mathcal{H}_S$ if $SH > 0$ and the process terminates.
   Otherwise, there are $r_2 = q - s - r_1 > 0$ irreducible diagonal blocks of order greater than 1 to be analyzed in the present Part 3.

2. Applying ModAH-algorithm, in turn, to each of the last r_2 blocks. Then:

   (a) If $\rho(|J_{B_{ii}}|) > 1$, then B_{ii} is not an $H-$matrix nor A by (2.5). In this case, NH take the value 2 and, by Theorem 2.2,

   - either $A \in {}_n\mathcal{H}^{\emptyset}$ provided $s = 0$,
   - or $A \in {}_n\mathcal{H}_S^0$ if $s \geq 1$.

   No more computations are needed and the conclusions in the Output of ABGH are correct.

   (b) If $\rho(|J_{B_{ii}}|) = 1$, then A has some diagonal block in $\mathcal{H}_M$. Then, if $s = 0$, put MH$=1$.

   (c) Otherwise, $\rho(|J_{B_{ii}}|) < 1$ and $B_{ii} \in \mathcal{H}_I$. (This result does not modify any final result.)

   (d) If NH remains null : go to the next i and repeat the process from 2.

3. When r_2 is exhausted, if NH remains null, each diagonal block of order greater than 1 has been analyzed and so, A is an $H-$matrix. Specifically,

   (i) If $s > 0$, $A \in \mathcal{H}_S$.

   (ii) Else, if MH$=1$, $A \in \mathcal{H}_M$.

   (iii) Else $A \in \mathcal{H}_I$.

The proof concludes noting that $\mathcal{H}_I \cup \mathcal{H}_M \cup \mathcal{H}_S \cup {}_n\mathcal{H}_N^0 \cup {}_n\mathcal{H}_S^0 \cup {}_n\mathcal{H}^{\emptyset}$ is a partition of the matrix set $\mathbb{C}^{n\times n}$ into the six mutually exclusive classes of $H-$ and non-$H-$matrices and, for any specific matrix A, one and only one of the six classes will be obtained. $\square$

Note that, in the theoretical case $A \in \mathcal{H}_M$ or whenever A is "very close" to belong to the class $\mathcal{H}_M$, the theorem may be of "little" practical value especially when n is relatively large. This is because the presence of round-off errors may lead to erroneous results and conclusions. So, in the practical implementation of ABGH we can modify Steps 6, 7 and 8 of ModAH-algorithm as

6. IF $m - 1 > TOL$: $r = m > 1$ ($F \in {}_n\mathcal{H}^{\emptyset}$), **END**
7. ELSE IF $1 - M > TOL$: $r = M < 1$ ($F \in \mathcal{H}_I$), **END**
8. ELSE IF $0 \leq 1 - m <$TOL and $0 \leq M - 1 <$TOL : $r = 1$ ($F \in \mathcal{H}_M$), **END**

with a tolerance bound TOL.

Note: Table 2 gives a detailed summary of what Theorem 4.1 states.

<table>
<tr><td colspan="6">$A \in \mathbb{C}^{n,n}$, $\mathcal{M}(A) = sI - B$, $s = \max_i |a_{ii}|$, $B \geq 0$. $\mathcal{F}(A) = [F_{ij}]$ is an Fnf of A.</td></tr>
<tr><td colspan="3">$H-$matrices ($\mathcal{H}$)</i></td><td colspan="3"><i>non-H$-$matrices (${}_n\mathcal{H}$)</td></tr>
<tr><td>$s > \rho(B)$</td><td colspan="2">$s = \rho(B)$</td><td colspan="3">$s < \rho(B)$</td></tr>
<tr><td>$A \in \mathcal{H}_I$</td><td>$A \in \mathcal{H}_S$</td><td>$A \in \mathcal{H}_M$</td><td colspan="2">$A \in {}_n\mathcal{H}^0$</td><td>$A \in {}_n\mathcal{H}^{\emptyset}$</td></tr>
<tr><td>A "invertible"</td><td>A "singular"</td><td>A "mixed"</td><td>$A \in {}_n\mathcal{H}_N^0$</td><td>$A \in {}_n\mathcal{H}_S^0$</td><td></td></tr>
<tr><td>Invertible</td><td>Reducible Singular</td><td></td><td></td><td>Reducible Singular</td><td></td></tr>
<tr><td>$\exists\ J_A$</td><td>$\nexists\ J_A$</td><td>$\exists\ J_A$</td><td colspan="2">$\nexists\ J_A$</td><td>$\exists\ J_A$</td></tr>
<tr><td>$\rho(|J_A|) < 1$</td><td></td><td>$\rho(|J_A|) = 1$</td><td></td><td></td><td>$\rho(|J_A|) > 1$</td></tr>
<tr><td>$a_{ii} \neq 0\ \forall i$</td><td>$\exists F_{ii} = [0]$</td><td>$a_{ii} \neq 0\ \forall i$</td><td>$\exists a_{ii} = 0$</td><td>$\exists F_{ii} = [0]$</td><td>$a_{ii} \neq 0\ \forall i$</td></tr>
<tr><td>$\rho(|J_{F_{ii}}|) < 1$ for all i</td><td>$\rho(|J_{F_{ii}}|) \leq 1$ for all $F_{ii} \neq [0]$</td><td>$\rho(|J_{F_{ii}}|) \leq 1$ for all i with equality holding for at least one index i</td><td>$\exists$ at least one irreducible diagonal block having singular main diagonal</td><td>$\exists J_{F_{ii}}$ for all $F_{ii} \neq [0]$ and some $\rho(|J_{F_{ii}}|) > 1$</td><td>$\rho(|J_{F_{ii}}|) > 1$ for at least one index i</td></tr>
</table>

Table 2: Classes of general $H-$ and non-$H-$matrices

5 Numerical Examples

In this section we give some *Numerical Examples* that cover all six classes of $H-$ and non-$H-$matrices of Table 2.

Example 8. *We begin with the matrix $\mathcal{F}(A) \in \mathbb{C}^{10\times 10}$ in (5.1) which is in its Fnf.*

The diagonal blocks F_{11}, F_{22}, F_{44} and F_{55} all belong to the class $\mathcal{H}_I$ as is readily checked. The two blocks F_{33} and F_{66} of orders 2×2 and 1×1, respectively, have one of their diagonal elements $(\mathcal{F}(A))_{55}$ and $(\mathcal{F}(A))_{10,10}$ given as x and y parameters. Playing with the values of x one can make the corresponding block belong to any of the four classes $\mathcal{H}_I$, $\mathcal{H}_M$, ${}_n\mathcal{H}^0$, ${}_n\mathcal{H}^{\emptyset}$, while the element y can take the value zero or not. So, by Theorem 4.1 and for a specific pair of values (x, y), we are in a position to know in advance the class to which $\mathcal{F}(A)$ belongs.

$$\mathcal{F}(A)=\left[\begin{array}{rrr|r|rr|r|rr|r}
3+4i & -1 & 2 & 0.1 & 0.4 & 0.7 & 1 & 1.3 & 1.6 & 1.9\\
0 & -3 & 0.5 & 0.1 & -0.3 & -0.4 & -0.5i & 1+2i & -2 & 0.4\\
4 & 3 & 6 & 1.1 & -1.2 & 1.5 & 2.7 & 3.8 & 0.7i & -1+2i\\
\hline
0 & 0 & 0 & i & 1+i & 1-i & 2 & 0.5 & 0.8 & -1\\
\hline
0 & 0 & 0 & 0 & x & 2 & 0.7 & 0.9 & 1.1 & 1.3i\\
0 & 0 & 0 & 0 & 3 & 8 & 2 & -1 & 3i & 7.5\\
\hline
0 & 0 & 0 & 0 & 0 & 0 & 1+i & 8 & 2.5 & -1.5\\
\hline
0 & 0 & 0 & 0 & 0 & 0 & 0 & 3 & 3 & i\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & -4.5 & 6 & 2-i\\
\hline
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & y
\end{array}\right]. \tag{5.1}$$

Consider now the permutation $\begin{pmatrix} 6 & 7 & 3 & 10 & 9 & 2 & 4 & 1 & 8 & 5 \end{pmatrix}$ that defines a permutation matrix P. Let $A = P\mathcal{F}(A)P^T$ from which

$$A=\begin{bmatrix}
8 & 2 & 0 & 7.5 & 3i & 0 & 0 & 0 & -1 & 3\\
0 & 1+i & 0 & -1.5 & 2.5 & 0 & 0 & 0 & 8 & 0\\
1.5 & 2.7 & 6 & -1+2i & 0.7i & 3 & 1.1 & 4 & 3.8 & -1.2\\
0 & 0 & 0 & y & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 2-i & 6 & 0 & 0 & 0 & -4.5 & 0\\
-0.4 & -0.5i & 0.5 & 0.4 & -2 & -3 & 0.1 & 0 & 1+2i & -0.3\\
1-i & 2 & 0 & -1 & 0.8 & 0 & i & 0 & 0.5 & 1+i\\
0.7 & 1 & 2 & 1.9 & 1.6 & -1 & 0.1 & 3+4i & 1.3 & 0.4\\
0 & 0 & 0 & i & 3 & 0 & 0 & 0 & 3 & 0\\
2 & 0.7 & 0 & 1.3i & 1.1 & 0 & 0 & 0 & 0.9 & x
\end{bmatrix}. \tag{5.2}$$

Giving (x, y) the pairs of values in Table 3 and running **ABGH** we find out that A belongs to the classes illustrated in Table 3, which correspond to the six subclasses of Table 2, respectively, as this was expected from the values of x and y given in the Fnf of A in (5.1).

(x,y)	$(-1,-0.1)$	$(2,0)$	$(-0.75,-0.1)$	$(0,1)$	$(-0.25,0)$	$(0.5,1)$
$A\in$	$\mathcal{H}_I$	$\mathcal{H}_S$	$\mathcal{H}_M$	${}_n\mathcal{H}_N^0$	${}_n\mathcal{H}_S^0$	${}_n\mathcal{H}^{\emptyset}$

Table 3: Classes of general $H-$ and non-$H-$matrices to which A belongs for various pairs (x, y).

6 Concluding Remarks

From the theory developed, the ABGH-Algorithm and the companion Theorem 4.1 it becomes crystal clear that the specific $H-$ or non-$H-$matrix class to which a given matrix $A \in \mathbb{C}^{n,n}$ belongs is fully justified.

As is known there exists in the literature Tarjan's Algorithm [12] which can efficiently replace the IRR- and BD-Algorithms provided one would modify it in order to incorporate Steps 2 and 7 of the IRR-Algorithm, then keep only the diagonal blocks of the produced Fnf and finally arrange the diagonal blocks in the way suggested by the BD-Algorithm.

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