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Eigenstructure of rank one updated matrices ☆

Rafael Bru, Rafael Cantó *, Ana M. Urbano

Institut de Matemàtica Multidisciplinar, Universitat Politècnica de València, 46071 València, Spain

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ABSTRACT

The relationship among eigenvalues of a given square matrix A and the rank one updated matrix $A + v_k q^*$, where v_k is an eigenvector of A associated with the eigenvalue λ_k and q is an arbitrary vector, was described by Brauer in 1952. In this work we study the relations between the Jordan structures of A and $A + v_k q^*$. More precisely, we analyze the generalized eigenvectors of the updated matrix in terms of the generalized eigenvectors of A, as well as the Jordan chains of the updated matrix. Further, we obtain similar results when we use a generalized eigenvector of A instead of the eigenvector v_k.

1. Introduction

The relationship between the Jordan structure of two matrices sufficiently close one to each other have been largely studied in the matrix literature. Boer and Thijsse [2] obtained relations for matrices with distinct eigenvalues. Furthermore, Marcus and Parilis [5] gave a complete description of the changes in the Jordan structure (counting

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* Corresponding author.
E-mail addresses: rbru@mat.upv.es (R. Bru), rcanto@mat.upv.es (R. Cantó), amurbano@mat.upv.es (A.M. Urbano).

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the length of the Jordan chains) of a perturbed matrix under small perturbations. Moro and Dopico [7] have researched the Jordan blocks of the complex matrix $A+B$ associated with an eigenvalue λ of A with geometric multiplicity g, when $\text{rank}(B) \leq g$. Beitia, Hoyos and Zaballa [1] studied the change of the Jordan structure of a perturbed matrix under small perturbations in one row of the given matrix. Recently Mehl, Mehrmann, Ran and Rodman [6] considered the perturbation theory of structured matrices under generic structured rank one perturbations, identifying generic Jordan structures of perturbed matrices.

In [4] the authors presented the relationships between the right and left eigenvectors of matrices A and $A+v_kq^T$, where v_k is an eigenvector of A. The same authors applied this result to deflation problems and to pole assignment for single-input single-output (SISO) and multi-input multi-output (MIMO) systems. In the MIMO systems defined by pairs (A, B) where A and B are $n \times n$ and $n \times m$ matrices, [4, Proposition 4.5] showed that the solution of the pole assignment is not unique in general. Moreover, it was possible to allocate poles even in the case of uncontrollable and unstable systems given by (A, B) and the closed-loop system $A+BF^T$ could be asymptotically stable with the feedback matrix F computed from the eigenvectors of A^T. Some connections between pole assignment and assignment of invariant factors on matrices with some prescribed submatrices were studied by Zaballa [11]. Concretely, by using the Hardy–Littlewood–Polya majorization relations and the interlacing inequalities for invariant factors, it was explained in [11, Theorem 2.6] when there exists an $m \times n$ matrix C with elements in a field such that $A+BC$ has n given monic polynomials as its invariant factors. This kind of applications motivate the study of problems involving Jordan structure as this work.

Thompson [10, Theorem 2] showed how similarity invariants of matrices with elements in a field change under a rank one update. In fact, Thompson obtained the interlacing inequalities for invariant factors of two square matrices A and $A + xy^T$ with elements in a field. This result does not say how the spectrum of $A + xy^T$ is obtained from the spectrum of A and the generalized eigenvectors of $A + xy^T$ from the given generalized eigenvectors of A. In general, this is a very difficult problem that we study in a specific case by using the well known Brauer Theorem [3]. This theorem gives the spectrum of the updated matrix $A + xy^T$ from the spectrum of the initial matrix A, where $x = v_k$ is an eigenvector of A associated with the eigenvalue λ_k and y is an arbitrary vector with elements in a field.

In this work we study the relations between the Jordan structures of the matrices $A \in \mathbb{C}^{n\times n}$ and $A + v_kq^* \in \mathbb{C}^{n\times n}$ using Brauer's Theorem. Concretely, we obtain the Jordan chains of $A + v_kq^*$ in terms of the given Jordan chains of A. Further, similar results when we use a generalized eigenvector of A instead of the eigenvector v_k are given.

Throughout this paper, we work with matrices $A \in \mathbb{C}^{n\times n}$ and denote by $\sigma(A) = \{\lambda_1, \lambda_2, \ldots, \lambda_s\}$ the spectrum of A with algebraic multiplicities $r_1, r_2, \ldots, r_s$. If $\{v_{it_i}, v_{i,t_i-1}, \ldots, v_{i2}, v_{i1}\}$ is a Jordan chain of A associated with λ_i of length t_i, then

$$\begin{cases} Av_{i1} = \lambda_i v_{i1} \\ Av_{ij} = \lambda_i v_{ij} + v_{i,j-1}, \qquad j = 2, 3, \ldots, t_i, \end{cases}$$

where

$$v_{ij} \in \ker(A - \lambda_i I)^j \setminus \ker(A - \lambda_i I)^{j-1}, \quad j = t_i, t_i - 1, \ldots, 1.$$

Suppose that A has k_i Jordan chains of length $t_{i_1}, t_{i_2}, \ldots, t_{i_{k_i}}$, for each eigenvalue λ_i, $i = 1, 2, \ldots, s$. Then the Jordan structure of A is

$$J_A = J(\lambda_1) \oplus J(\lambda_2) \oplus \cdots \oplus J(\lambda_s)$$

where

$$J(\lambda_i) = J_{t_{i_1}}(\lambda_i) \oplus J_{t_{i_2}}(\lambda_i) \oplus \cdots \oplus J_{t_{i_{k_i}}}(\lambda_i)$$

and

$$J_{t_i}(\lambda_i) = \begin{bmatrix} \lambda_i & 1 & \ldots & 0 & 0 \\ 0 & \lambda_i & \ldots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \ldots & \lambda_i & 1 \\ 0 & 0 & \ldots & 0 & \lambda_i \end{bmatrix}_{t_i \times t_i}.$$

Without loss of generality, we apply Brauer's Theorem with the eigenvalue λ_1 of A and the associated eigenvector v_1. Then the updated matrix $A + v_1 q^*$ has v_1 as an eigenvector associated with the new eigenvalue $\mu = \lambda_1 + q^* v_1$. Moreover, $\sigma(A + v_1 q^*) = \{\mu, \lambda_1, \lambda_2, \ldots, \lambda_s\}$ with algebraic multiplicities 1, $r_1 - 1$, r_2, …, r_s when $\mu \notin \sigma(A)$, or with the corresponding multiplicities if $\mu \in \sigma(A)$.

We study the changes of the Jordan structure when $\mu \notin \sigma(A)$ in Section 2 and when $\mu \in \sigma(A)$ in Section 3. To do this, we construct the Jordan chains of each eigenvalue of the updated matrix from the Jordan chains of the corresponding eigenvalue of the initial matrix, and we call the Jordan chain $\{w_{it_i}, w_{i,t_i-1}, \ldots, w_{i2}, w_{i1}\}$ of $A + v_1 q^*$ the *updated Jordan chain of* $\{v_{it_i}, v_{i,t_i-1}, \ldots, v_{i2}, v_{i1}\}$ of A.

Finally, in Section 4 we study how the Jordan chains of A are changing when we apply the rank one perturbation $v_{1j} q^*$, where v_{1j} is a generalized eigenvector of A associated with λ_1 instead of its eigenvector v_1 and q is an n-dimensional vector such that $q^* v_{1h} = 0$, $h \leq j$.

2. Jordan structure when $\mu \notin \sigma(A)$

In this section we study the changes of the Jordan structure of a matrix A when the new eigenvalue $\mu = \lambda_1 + q^* v_1 \notin \sigma(A)$. By Brauer's Theorem (see [4]), v_1 is the eigenvector associated with μ of the updated matrix $A + v_1 q^*$. The updated Jordan chains are obtained in the following results. In Theorem 1 we prove that the Jordan chains of A and $A + v_1 q^*$ associated with λ_i have the same length, for $i = 2, 3, \dots, s$. In Theorem 2, the Jordan structure of $A + v_1 q^*$ associated with λ_1 is given.

Theorem 1 *(Structure associated with $\lambda_i \neq \lambda_1$). Let $A \in \mathbb{C}^{n\times n}$. Consider an eigenvalue $\lambda_i \neq \lambda_1$ and v_1 an eigenvector of A associated with λ_1. Let $\{v_{it_i}, v_{i,t_i-1}, \dots, v_{i2}, v_{i1}\}$ be a Jordan chain of A associated with λ_i. Let q be an n-dimensional vector such that $\mu = \lambda_1 + q^* v_1 \notin \sigma(A)$. Then $A + v_1 q^*$ has the updated Jordan chain $\{w_{it_i}, w_{i,t_i-1}, \dots, w_{i2}, w_{i1}\}$, associated with the eigenvalue λ_i, where*

$$w_{ij} = v_{ij} - \sum_{s=1}^{j} \frac{q^* v_{is}}{(\mu - \lambda_i)^{j+1-s}}\, v_1, \quad j = 1, 2, \dots, t_i.$$

Proof. Since $\{v_{it_i}, v_{i,t_i-1}, \dots, v_{i2}, v_{i1}\}$ is a Jordan chain associated with the eigenvalue λ_i we have

$$\begin{aligned} Av_{ij} &= \lambda_i v_{ij} + v_{i,j-1}, \qquad j = t_i, t_i - 1, \dots, 2, \\ Av_{i1} &= \lambda_i v_{i1}. \end{aligned}$$

It remains to prove

$$\begin{aligned} (A + v_1 q^*) w_{ij} &= \lambda_i w_{ij} + w_{i,j-1}, \qquad j = t_i, t_i - 1, \dots, 2, \\ (A + v_1 q^*) w_{i1} &= \lambda_i w_{i1}. \end{aligned}$$

For $j = t_i, t_i - 1, \dots, 2$, we have

$$\begin{aligned} (A + v_1 q^*) w_{ij} &= (A + v_1 q^*) \left(v_{ij} - \sum_{s=1}^{j} \frac{q^* v_{is}}{(\mu - \lambda_i)^{j+1-s}}\, v_1 \right) \\ &= Av_{ij} + (q^* v_{ij}) v_1 - \sum_{s=1}^{j} \frac{q^* v_{is}}{(\mu - \lambda_i)^{j+1-s}}\, (A + v_1 q^*) v_1 \\ &= \lambda_i v_{ij} + v_{i,j-1} + (q^* v_{ij}) v_1 - \sum_{s=1}^{j} \frac{q^* v_{is}}{(\mu - \lambda_i)^{j+1-s}}\, \mu v_1 \\ &\quad + \sum_{s=1}^{j} \frac{q^* v_{is}}{(\mu - \lambda_i)^{j+1-s}}\, \lambda_i v_1 - \sum_{s=1}^{j} \frac{q^* v_{is}}{(\mu - \lambda_i)^{j+1-s}}\, \lambda_i v_1 \\ &= \lambda_i \left(v_{ij} - \sum_{s=1}^{j} \frac{q^* v_{is}}{(\mu - \lambda_i)^{j+1-s}}\, v_1 \right) + v_{i,j-1} + (q^* v_{ij}) v_1 \end{aligned}$$

$$
\begin{aligned}
&- \sum_{s=1}^{j} \frac{q^* v_{is}}{(\mu - \lambda_i)^{j+1-s}} (\mu - \lambda_i) v_1 \\
&= \lambda_i \left(v_{ij} - \sum_{s=1}^{j} \frac{q^* v_{is}}{(\mu - \lambda_i)^{j+1-s}} v_1 \right) + \left(v_{i,j-1} - \sum_{s=1}^{j-1} \frac{q^* v_{is}}{(\mu - \lambda_i)^{j-s}} v_1 \right) \\
&= \lambda_i w_{ij} + w_{i,j-1}.
\end{aligned}
$$

Finally, for $j = 1$, the expression of the eigenvector associated with λ_i

$$
w_{i1} = v_{i1} - \frac{q^* v_{i1}}{\mu - \lambda_i} v_1,
$$

is proved in [8] (see also [4, Proposition 1.1.]).

As a consequence, we conclude that $\{w_{it_i}, w_{i,t_i-1}, \ldots, w_{i2}, w_{i1}\}$ is the updated Jordan chain of the matrix $A + v_1 q^*$ associated with λ_i. □

Theorem 2 *(Structure associated with λ_1). Let $A \in \mathbb{C}^{n\times n}$ and let v_1 be an eigenvector of A associated with λ_1. Let $\{v_{1t_1}, v_{1,t_1-1}, \ldots, v_{12}, v_{11}\}$ be a Jordan chain of A associated with λ_1 of length t_1. Let q be an n-dimensional vector such that $\mu = \lambda_1 + q^* v_1 \notin \sigma(A)$. Then $A + v_1 q^*$ has the updated Jordan chain $\{w_{1d_1}, w_{1,d_1-1}, \ldots, w_{12}, w_{11}\}$ of length d_1, associated with λ_1, as follows*

(1) If $v_{11} \neq v_1$, then $d_1 = t_1$ and

$$
w_{1j} = v_{1j} - \sum_{s=1}^{j} \frac{q^* v_{1s}}{(\mu - \lambda_i)^{j+1-s}} v_1, \quad j = 1, 2, \ldots, t_1.
$$

(2) If $v_{11} = v_1$ then $d_1 = t_1 - 1$. We distinguish two cases:

(2.1) If $t_1 = 1$, there is not any updated Jordan chain of $\{v_1\}$ associated with λ_1 in $A + v_1 q^*$. Furthermore, v_1 becomes an eigenvector of $A + v_1 q^*$ associated with μ.

(2.2) If $t_1 > 1$, then

$$
\begin{aligned}
w_{11} &= v_{12} - \frac{1 + q^* v_{12}}{\mu - \lambda_1} v_1, \\
w_{1j} &= v_{1,j+1} - \sum_{s=3}^{j+1} \frac{q^* v_{1s}}{(\mu - \lambda_1)^{j+2-s}} v_1 - \frac{1 + q^* v_{12}}{(\mu - \lambda_1)^j} v_1,
\end{aligned}
$$

for $j = 2, 3, \ldots, t_1 - 1$.

Proof.

(1) This case can be proved analogously to Theorem 1.

(2) For $v_{11} = v_1$, we consider two cases:

(2.1) The first part of the assertion is clear. The second part is included in Brauer's Theorem.

(2.2) For $j = t_1 - 1, t_1 - 2, \ldots, 2$, we have

$$\begin{aligned}
&(A + v_1 q^*)(w_{1j}) = \\
&= (A + v_1 q^*)\left(v_{1,j+1} - \sum_{s=3}^{j+1} \frac{q^* v_{1s}}{(\mu - \lambda_1)^{j+2-s}}\, v_1 - \frac{1 + q^* v_{12}}{(\mu - \lambda_1)^j}\, v_1\right) \\
&= A v_{1,j+1} + (q^* v_{1,j+1}) v_1 - \sum_{s=3}^{j+1} \frac{q^* v_{1s}}{(\mu - \lambda_1)^{j+2-s}}\, \mu v_1 - \frac{1 + q^* v_{12}}{(\mu - \lambda_1)^j}\, \mu v_1 \\
&= \left(\lambda_1 v_{1,j+1} - \lambda_1 \sum_{s=3}^{j+1} \frac{q^* v_{1s}}{(\mu - \lambda_1)^{j+2-s}}\, v_1 - \lambda_1 \frac{1 + q^* v_{12}}{(\mu - \lambda_1)^j}\, v_1\right) \\
&\quad + v_{1j} + (q^* v_{1,j+1}) v_1 - \sum_{s=3}^{j+1} \frac{q^* v_{1s}}{(\mu - \lambda_1)^{j+2-s}}\, (\mu - \lambda_1) v_1 \\
&\quad - \frac{1 + q^* v_{12}}{(\mu - \lambda_1)^j}\, (\mu - \lambda_1) v_1 \\
&= \lambda_1 \left(v_{1,j+1} - \sum_{s=3}^{j+1} \frac{q^* v_{1s}}{(\mu - \lambda_1)^{j+2-s}}\, v_1 - \frac{1 + q^* v_{12}}{(\mu - \lambda_1)^j}\, v_1\right) \\
&\quad + \left(v_{1j} - \sum_{s=3}^{j} \frac{q^* v_{1s}}{(\mu - \lambda_1)^{j+1-s}}\, v_1 - \frac{1 + q^* v_{12}}{(\mu - \lambda_1)^{j-1}}\, v_1\right) = \lambda_1 w_{1j} + w_{i,j-1}.
\end{aligned}$$

As v_1 is the eigenvector of the matrix $A + v_1 q^*$, associated with μ, then

$$\begin{aligned}
(A + v_1 q^*) w_{11} &= (A + v_1 q^*)\left(v_{12} - \frac{1 + q^* v_{12}}{\mu - \lambda_1}\, v_1\right) \\
&= A v_{12} + (q^* v_{12}) v_1 - \frac{1 + q^* v_{12}}{\mu - \lambda_1}\, \mu v_1 \\
&= \lambda_1 v_{12} + v_1 + (q^* v_{12}) v_1 - \frac{1 + q^* v_{12}}{\mu - \lambda_1}\, \mu v_1 \\
&= \lambda_1 v_{12} + \left\{1 + q^* v_{12} - \frac{1 + q^* v_{12}}{\mu - \lambda_1}\, \mu\right\} v_1 \\
&= \lambda_1 v_{12} - \lambda_1 \frac{1 + q^* v_{12}}{\mu - \lambda_1}\, v_1 = \lambda_1 \left(v_{12} - \frac{1 + q^* v_{12}}{\mu - \lambda_1}\, v_1\right) = \lambda_1 w_{11},
\end{aligned}$$

that is, w_{11} is the eigenvector of $A + v_1 q^*$ associated with λ_1 in this chain. Therefore, $\{w_{1,t_1-1}, w_{1,t_1-2}, \ldots, w_{11}\}$ is the updated Jordan chain of length $t_1 - 1$, of the matrix $A + v_1 q^*$, associated with λ_1. $\square$

Example 1. Consider the following matrix in Jordan form

$$A = J_3(1) \oplus J_2(1) \oplus J_2(2) \oplus J_1(3)$$

whose eigenvalues are $\lambda_1 = 1$, $\lambda_2 = 2$ and $\lambda_3 = 3$. The Jordan chains are: $\{e_3, e_2, e_1\}$ and $\{e_5, e_4\}$ associate with $\lambda_1 = 1$, $\{e_7, e_6\}$ associate with $\lambda_2 = 2$ and $\{e_8\}$ associate with $\lambda_3 = 3$.

Consider the eigenvector e_1 and the vector $q^* = [q_1\ q_2\ q_3\ q_4\ q_5\ q_6\ q_7\ q_8]$, such that $q_1 \notin \{0, 1, 2\}$. Then $1 + q^*e_1 = 1 + q_1 \notin \sigma(A)$ and the eigenvalues of $A + e_1q^*$ are $\mu = 1 + q_1$, $\lambda_1 = 1$, $\lambda_2 = 2$ and $\lambda_3 = 3$. The Jordan chains of this matrix are:

Jordan chains of A	Updated Jordan chains of $A + e_1q^*$
Eigenvalue $\lambda_2 = 2$ $\{e_7,\ e_6\}$	Eigenvalue $\lambda_2 = 2$ $\left\{e_7 - \left(\frac{q_6}{(q_1-1)^2} + \frac{q_7}{q_1-1}\right) e_1,\ e_6 - \frac{q_6}{q_1-1}\, e_1\right\}$
Eigenvalue $\lambda_3 = 3$ $\{e_8\}$	Eigenvalue $\lambda_3 = 3$ $\left\{e_8 - \frac{q_8}{q_1-1}\, e_1\right\}$

Jordan chains of A	Updated Jordan chains of $A + e_1q^*$
Eigenvalue $\lambda_1 = 1$ $\{e_5,\ e_4\}$	Eigenvalue $\lambda_1 = 1$ $\left\{e_5 - \left(\frac{q_4}{(q_1-1)^2} + \frac{q_5}{q_1-1}\right) e_1,\ e_4 - \frac{q_4}{q_1-1}\, e_1\right\}$
Eigenvalue $\lambda_1 = 1$ $\{e_3,\ e_2,\ e_1\}$	Eigenvalue $\lambda_1 = 1$ $\left\{e_3 - \left(\frac{q_3}{q_1} + \frac{1+q_2}{q_1^2}\right) e_1,\ e_2 - \frac{1+q_2}{q_1}\, e_1\right\}$ Eigenvalue $\mu = 1 + q_1$ $\{e_1\}$

The invariant factors of matrices A and $A + e_1q^*$ are

$$S_8(A) = (\lambda - 1)^3(\lambda - 2)^2(\lambda - 3) \quad S_7(A) = (\lambda - 1)^2,$$
$$S_6(A) = S_5(A) = \cdots = S_1(A) = 1$$

and

$$S_8(A + e_1q^*) = (\lambda - 1)^2(\lambda - 2)^2(\lambda - 3)(\lambda - (1 + q_1))$$
$$S_7(A + e_1q^*) = (\lambda - 1)^2$$
$$S_6(A + e_1q^*) = S_5(A + e_1q^*) = \cdots = S_1(A + e_1q^*) = 1.$$

Note that Thompson [10, Theorem 2] holds.

3. Jordan structure when $\mu \in \sigma(A)$

Now, we study the changes of the Jordan structure of a matrix A when the new eigenvalue $\mu = \lambda_1 + q^* v_1$ belongs to $\sigma(A)$. We consider two possibilities: first when μ is equal to some eigenvalue λ_i, for $i = 2, 3, \dots, s$. In this case, without loss of generality, we suppose that $\mu = \lambda_2$. Then to get the updated Jordan chains of $A + v_1 q^*$, we work with Jordan chains associated with the λ_i, for $i = 3, 4, \dots, s$, with λ_1 and with λ_2. The second possibility is when $\mu = \lambda_1$.

3.1. Jordan structure when $\mu = \lambda_2$

We can obtain the updated Jordan chains of $A+v_1 q^*$ associated with λ_i, $i = 3, 4, \dots, s$, applying Theorem 1 and associated with λ_1 applying Theorem 2. To study the updated Jordan chains associated with λ_2 we consider two cases:

1. All Jordan chains of A associated with λ_2 satisfy that the inner product of the vector q with the eigenvector of the corresponding chain is zero (case (1) in the next theorem).
2. Otherwise, there exists Jordan chains of A associated with λ_2 such that the inner product of the vector q with the corresponding eigenvector is nonzero (case (2) in the next theorem).

Theorem 3 *(Structure associated with λ_2). Let $A \in \mathbb{C}^{n\times n}$. Consider the eigenvalue $\lambda_2 \neq \lambda_1$ and v_1 an eigenvector of A associated with λ_1. Suppose that A has k_2 Jordan chains of length $t_{2_1} \le t_{2_2} \le \cdots \le t_{2_{k_2}}$, denoted by $\left\{v^{(g)}_{2t_{2_g}}, v^{(g)}_{2,t_{2_g}-1}, \dots, v^{(g)}_{22}, v^{(g)}_{21}\right\}$, with $g = 1, 2, \dots, k_2$. Let q be an n-dimensional vector such that $\mu = \lambda_1 + q^* v_1 = \lambda_2$. Then the length and number of Jordan chains associated with λ_2 of the updated matrix $A + v_1 q^*$ are:*

(1) If $q^* v^{(g)}_{21} = 0$, for $g = 1, 2, 3, \dots, k_2$, the updated matrix $A + v_1 q^*$ has $k_2 + 1$ Jordan chains of length $t_{2_1}, t_{2_2}, \dots, t_{2_{k_2}}$, 1, defined as follows

(1.1) For $g = 1, 2, \dots, k_2$, $\left\{w^{(g)}_{2t_{2_g}}, w^{(g)}_{2,t_{2_g}-1}, \dots, w^{(g)}_{22}, w^{(g)}_{21}\right\}$ is the updated Jordan chain of $\left\{v^{(g)}_{2t_{2_g}}, v^{(g)}_{2,t_{2_g}-1}, \dots, v^{(g)}_{22}, v^{(g)}_{21}\right\}$, where

$$\begin{aligned} w^{(g)}_{2t_{2_g}} &= v^{(g)}_{2t_{2_g}} \\ w^{(g)}_{2l} &= v^{(g)}_{2l} + (q^* v^{(g)}_{2,l+1}) v_1, \quad l = t_{2_g} - 1, t_{2_g} - 2, \dots, 2, 1 \end{aligned}$$

(1.2) $\{v_1\}$ is the new Jordan chain of length 1.

(2) There exists $j_1, j_2, \dots, j_p$, such that $t_{2_1} \le t_{2_{j_1}} \le t_{2_{j_2}} \le \cdots \le t_{2_{j_p}} \le t_{2_{k_2}}$ and $q^* v^{(j_i)}_{21} \neq 0$, for $i = 1, 2, \dots, p$. Then $A + v_1 q^*$ has k_2 Jordan chains of length $t_{2_1}, \dots, t_{2_{j_{p-1}}}, t_{2_{j_p}} + 1, t_{2_{j_{p+1}}}, \dots, t_{2_{k_2}}$, defined as follows

(2.1) $\left\{w^{(j_p)}_{2,t_{2_{j_p}}+1}, w^{(j_p)}_{2t_{2_{j_p}}}, \ldots, w^{(j_p)}_{22}, w^{(j_p)}_{21}\right\}$*, of length* $t_{2_{j_p}} + 1$*, is the updated Jordan chain of* $\left\{v^{(j_p)}_{2t_{2_{j_p}}}, v^{(j_p)}_{2,t_{2_{j_p}}-1}, \ldots, v^{(j_p)}_{22}, v^{(j_p)}_{21}\right\}$*, where*

$$\begin{aligned} w^{(j_p)}_{2,t_{2_{j_p}}+1} &= \frac{1}{q^* v^{(j_p)}_{21}} v^{(j_p)}_{2t_{2_{j_p}}} \\ w^{(j_p)}_{2,l+1} &= \frac{1}{q^* v^{(j_p)}_{21}} v^{(j_p)}_{2l} + \frac{q^* v^{(j_p)}_{2,l+1}}{q^* v^{(j_p)}_{21}} v_1, \quad l = t_{2_{j_p}} - 1, \ldots, 2, 1 \\ w^{(j_p)}_{21} &= v_1 \end{aligned}$$

(2.2) *For* $i = 1, 2, \ldots, p-1$, $\left\{w^{(j_i)}_{2,t_{2_{j_i}}}, w^{(j_i)}_{2t_{2_{j_i}-1}}, \ldots, w^{(j_i)}_{22}, w^{(j_i)}_{21}\right\}$ *is the updated Jordan chain of* $\left\{v^{(j_i)}_{2t_{2_{j_i}}}, v^{(j_i)}_{2,t_{2_{j_i}}-1}, \ldots, v^{(j_i)}_{22}, v^{(j_i)}_{21}\right\}$*, where*

$$\begin{aligned} w^{(j_i)}_{2,t_{2_{j_i}}} &= v^{(j_i)}_{2,t_{2_{j_i}}} - \frac{q^* v^{(j_i)}_{21}}{q^* v^{(j_p)}_{21}} v^{(j_p)}_{2t_{2_{j_p}}} \\ w^{(j_i)}_{2l} &= v^{(j_i)}_{2l} - \frac{q^* v^{(j_i)}_{21}}{q^* v^{(j_p)}_{21}} v^{(j_p)}_{2l} + \left(q^* v^{(j_i)}_{2,l+1} - \frac{q^* v^{(j_i)}_{21}}{q^* v^{(j_p)}_{21}} q^* v^{(j_p)}_{2,j+1} \right) v_1, \\ l &= t_{2_{j_i}} - 1, \ldots, 2, 1 \end{aligned}$$

(2.3) *For* $1 \leq g \leq k_2$*, such that* $g \notin \{j_1, j_2, \ldots, j_p\}$*, it is satisfies that* $q^* v^{(j_i)}_{21} \neq 0$*. Then* $\left\{w^{(g)}_{2t_{2_g}}, w^{(g)}_{2,t_{2_g}-1}, \ldots, w^{(g)}_{22}, w^{(g)}_{21}\right\}$*, that it is the updated Jordan chain of* $\left\{v^{(g)}_{2t_{2_g}}, v^{(g)}_{2,t_{2_g}-1}, \ldots, v^{(g)}_{22}, v^{(g)}_{21}\right\}$*, is defined in case (1.1).*

Proof.

(1) If $q^* v^{(g)}_{21} = 0$, for $g = 1, 2, 3, \ldots, k_2$, we have

(1.1) For $g = 1, 2, 3, \ldots, k_2$,

$$\begin{aligned} (A + v_1 q^*) w^{(g)}_{2t_{2_g}} = (A + v_1 q^*) v^{(g)}_{2t_{2_g}} &= \lambda_2 v^{(g)}_{2t_{2_g}} + v^{(g)}_{2t_{2_g}-1} + (q^* v^{(g)}_{2t_{2_g}}) v_1 \\ &= \lambda_2 w^{(g)}_{2t_{2_g}} + w^{(g)}_{2t_{2_g}-1}. \end{aligned}$$

Further, for $l = t_{2_g} - 1, t_{2_g} - 2, \ldots, 2$,

$$\begin{aligned} (A + v_1 q^*) w^{(g)}_{2l} &= (A + v_1 q^*)(v^{(g)}_{2l} + (q^* v^{(g)}_{2,l+1}) v_1) \\ &= \lambda_2 v^{(g)}_{2l} + v^{(g)}_{2,l-1} + (q^* v^{(g)}_{2l}) v_1 + \lambda_2 (q^* v^{(g)}_{2,l+1}) v_1 \\ &= \lambda_2 (v^{(g)}_{2l} + (q^* v^{(g)}_{2,l+1}) v_1) + (v^{(g)}_{2,l-1} + (q^* v^{(g)}_{2l}) v_1) \\ &= \lambda_2 w^{(g)}_{2l} + w^{(g)}_{2,l-1} \end{aligned}$$

and finally

$$
\begin{aligned}
(A + v_1 q^*) w_{21}^{(g)} &= (A + v_1 q^*)(v_{21}^{(g)} + (q^* v_{22}^{(g)}) v_1) \\
&= \lambda_2 (v_{21}^{(g)} + (q^* v_{22})^{(g)} v_1)) = \lambda_2 w_{21}^{(g)}.
\end{aligned}
$$

Therefore, $\left\{w_{2t_{2_g}}^{(g)}, w_{2,t_{2_g}-1}^{(g)}, \ldots, w_{22}^{(g)}, w_{21}^{(g)}\right\}$ is the updated Jordan chain of $\left\{v_{2t_{2_g}}^{(g)}, v_{2,t_{2_g}-1}^{(g)}, \ldots, v_{22}^{(g)}, v_{21}^{(g)}\right\}$, of length t_{2_g}, associated with λ_2.

(1.2) Since

$$
(A + v_1 q^*)\, v_1 = A v_1 + (q^* v_1) v_1 = (\lambda_1 + q^* v_1) v_1 = \lambda_2 v_1
$$

$\{v_1\}$ is the new Jordan chain of length 1 associated with λ_2.

(2) The results can be proved with the techniques used before. □

Example 2. Consider the matrix $A = J_3(1) \oplus J_2(2) \oplus J_2(2) \oplus J_2(2)$, whose Jordan chains are: $\left\{v_{13}^{(1)}, v_{12}^{(1)}, v_{11}^{(1)}\right\} = \{e_3, e_2, e_1\}$ associated with $\lambda_1 = 1$, $\left\{v_{22}^{(1)}, v_{21}^{(1)}\right\} = \{e_5, e_4\}$, $\left\{v_{22}^{(2)}, v_{21}^{(2)}\right\} = \{e_7, e_6\}$ and $\left\{v_{22}^{(3)}, v_{21}^{(3)}\right\} = \{e_9, e_8\}$ associated with $\lambda_2 = 2$. If we take the vector $q^* = [1\ q_2\ q_3\ q_4\ q_5\ q_6\ q_7\ q_8\ q_9]$ then $\mu = 1 + q^* e_1 = 2 \in \sigma(A)$ and the eigenvalues of $A + e_1 q^*$ are $\lambda_1 = 1$ and $\lambda_2 = 2$. The length of the Jordan chains of $A + e_1 q^*$ depend on the values of q_4, q_6 and q_8. If q_4 and q_6 are not zero and $q_8 = 0$, and the Jordan chains of $A + e_1 q^*$ are:

- For $\lambda_1 = 1$ by Theorem 2 and taking into account that $q_1 = 1$, the updated Jordan chain of $\{e_3, e_2, e_1\}$ is $\{e_3 - (q_3 + (1 + q_2))\, e_1,\ e_2 - (1 + q_2) e_1\}$.
- For $\lambda_2 = 2$ by Theorem 3 the updated Jordan chains are:

Jordan chains of A	Updated Jordan chains of $A + e_1 q^*$
$\{e_9,\ e_8\}$ $q^* e_8 = q_8 = 0$	$\{e_9,\ e_8 + q_9 e_1\}$
$\{e_5,\ e_4\}$ $q^* e_4 = q_4 \neq 0$ chain of maximal length	$\left\{\frac{1}{q_4} e_5,\ \frac{1}{q_4} e_4 + \frac{q_5}{q_4} e_1,\ e_1\right\}$
$\{e_7,\ e_6\}$ $q^* e_6 = q_6 \neq 0$	$\left\{e_7 - \frac{q_6}{q_4} e_5, e_6 - \frac{q_6}{q_4} e_4 + \left(q_7 - \frac{q_6}{q_4} q_5\right) e_1\right\}$

3.2. Jordan structure when $\mu = \lambda_1$

Now, we study the case $\mu = \lambda_1$, that is when no eigenvalue is changed. Note that, $q^* v_1 = 0$. Since we do not change any eigenvalue, it seems nothing is done. However, we can have a new matrix, with some special property, with the same spectrum as the first one but with a new Jordan structure. In this case we obtain the updated Jordan chains of $A + v_1 q^*$ associated with λ_i, $i = 2, 3, \ldots, s$, applying Theorem 1 with $\mu = \lambda_1$. The Jordan structure of $A + v_1 q^*$ associated with λ_1 is given in the following theorem.

Theorem 4 *(Structure associated with λ_1). Let $A \in \mathbb{C}^{n\times n}$. Consider the eigenvalue λ_1 and the associated eigenvector v_1. Suppose that A has k_1 Jordan chains of length $t_{1_1}, t_{1_2}, \ldots, t_{1_{k_1}}$ associated with λ_1, denoted by $\left\{v^{(g)}_{1t_{1_g}},\ v^{(g)}_{1,t_{1_g}-1}, \ldots, v^{(g)}_{12}, v^{(g)}_{11}\right\}$, with $g = 1, 2, \ldots, k_1$ and $v^{(1)}_{11} = v_1$. Let q be an n-dimensional vector such that $q^* v_1 = 0$. Then the length and number of Jordan chains associated with λ_1 of the updated matrix $A + v_1 q^*$ are:*

(1) If $q^* v^{(g)}_{11} = 0$, for $g = 1, 2, 3, \ldots, k_1$, the updated matrix $A + v_1 q^*$ has k_1 Jordan chains of length $t_{1_1}, t_{1_2}, \ldots, t_{1_{k_1}}$, defined as follows

(1.1) $\left\{w^{(1)}_{1t_{1_1}}, w^{(1)}_{1,t_{1_1}-1}, \ldots, w^{(1)}_{12}, w^{(1)}_{11}\right\}$ of length t_{1_1}, is the updated Jordan chain of $\left\{v^{(1)}_{1t_{1_1}}, v^{(1)}_{1,t_{1_1}-1}, \ldots, v^{(1)}_{12}, v^{(1)}_{11} = v_1\right\}$, where

$$
\begin{aligned}
w^{(1)}_{11} &= v_1 \\
w^{(1)}_{1t_{1_1}} &= \frac{1}{1 + q^* v^{(1)}_{12}}\ v^{(1)}_{1t_{1_1}}, \\
w^{(1)}_{1l} &= \frac{1}{1 + q^* v^{(1)}_{12}}\ v^{(1)}_{1l} + \frac{q^* v^{(1)}_{1,l+1}}{1 + q^* v^{(1)}_{12}}\ v_1, \quad l = t_{1_1} - 1, \ldots, 3, 2,
\end{aligned}
$$

(1.2) For $g = 2, \ldots, k_1$, $\left\{w^{(g)}_{1t_{1_g}}, w^{(g)}_{1,t_{1_g}-1}, \ldots, w^{(g)}_{12}, w^{(g)}_{11}\right\}$ is the updated Jordan chain of $\left\{v^{(g)}_{1t_{1_g}}, v^{(g)}_{1,t_{1_g}-1}, \ldots, v^{(g)}_{12}, v^{(g)}_{11}\right\}$ where

$$
\begin{aligned}
w^{(g)}_{1t_{1_g}} &= v^{(g)}_{1t_{1_g}}, \\
w^{(g)}_{1l} &= v^{(g)}_{1l} + (q^* v^{(g)}_{1,l+1}) v_1, \quad l = t_{1_g} - 1, t_{1_g} - 2, \ldots, 2, 1
\end{aligned}
$$

Note that, in this case A and $A+v_1 q^*$ are similar because they have the same Jordan structure with different generalized eigenvectors.

(2) If there exists a unique index j, $2 \le j \le k_1$, such that $q^* v^{(j)}_{11} \neq 0$, we have

(2.1) If $t_{1_1} = 1$ the updated matrix $A+v_1 q^*$ has the following $k_1 - 1$ Jordan chains of length $t_{1_2}, t_{1_3} \ldots, t_{1_j} + 1, t_{1_{j+1}}, \ldots, t_{1_{k_1}}$,

(2.1.1) $\left\{w^{(j)}_{1,t_{1_j}+1}, w^{(j)}_{1t_{1_j}}, \ldots, w^{(j)}_{12}, w^{(j)}_{11}\right\}$, of length $t_{1_j}+1$, is the updated Jordan chain of $\left\{v^{(j)}_{1t_{1_j}}, v^{(j)}_{1,t_{1_j}-1}, \ldots, v^{(j)}_{12}, v^{(j)}_{11}\right\}$ where

$$w^{(j)}_{1,t_{1_j}+1} = \frac{1}{q^* v^{(j)}_{11}} \; v^{(j)}_{1t_{1_j}}$$

$$w^{(j)}_{1,l+1} = \frac{1}{q^* v^{(j)}_{11}} \; v^{(j)}_{1l} + \frac{q^* v^{(j)}_{1,l+1}}{q^* v^{(j)}_{11}} \; v_1, \quad l = t_{1_j} - 1, \ldots, 2, 1,$$

$$w^{(j)}_{11} = v_1$$

(2.1.2) *For* $g = 2, 3, \ldots, k_1$, $g \neq j$, $\left\{w^{(g)}_{1,t_{1_g}}, w^{(g)}_{1t_{1_g}}, \ldots, w^{(g)}_{12}, w^{(g)}_{11}\right\}$ *is the updated Jordan chain of* $\left\{v^{(g)}_{1t_{1_g}}, v^{(g)}_{1,t_{1_g}-1}, \ldots, v^{(g)}_{12}, v^{(g)}_{11}\right\}$ *and it is defined as in case (1.2).*

(2.2) *If* $t_{1_1} > 1$, *we have two cases:*

(2.2.1) $t_{1_1} > t_{1_j}$, *then the updated matrix* $A + v_1 q^*$ *has the following* k_1 *Jordan chains of length* $t_{1_1}, t_{1_2} \ldots, t_{1_j}, t_{1_{j+1}}, \ldots, t_{1_{k_1}}$,

(2.2.1.1) $\left\{w^{(1)}_{1,t_{1_1}}, w^{(1)}_{1t_{1_1}-1}, \ldots, w^{(1)}_{12}, w^{(1)}_{11}\right\}$ *is the updated Jordan chain of* $\left\{v^{(1)}_{1t_{1_1}}, v^{(1)}_{1,t_{1_1}-1}, \ldots, v^{(1)}_{12}, v^{(1)}_{11} = v_1\right\}$ *and it is defined as in case (1.1).*

(2.2.1.2) *For* $g = 2, 3, \ldots, k_1$, $g \neq j$, $\left\{w^{(g)}_{1,t_{1_g}}, w^{(g)}_{1t_{1_g}}, \ldots, w^{(g)}_{12}, w^{(g)}_{11}\right\}$ *is the updated Jordan chain of* $\left\{v^{(g)}_{1t_{1_g}}, v^{(g)}_{1,t_{1_g}-1}, \ldots, v^{(g)}_{12}, v^{(g)}_{11}\right\}$ *and it is defined as in case (1.2).*

(2.2.1.3) $\left\{w^{(j)}_{1,t_{1_j}}, w^{(j)}_{1t_{1_j}-1}, \ldots, w^{(j)}_{12}, w^{(j)}_{11}\right\}$ *is the updated Jordan chain of* $\left\{v^{(j)}_{1t_{1_j}}, v^{(j)}_{1,t_{1_j}-1}, \ldots, v^{(j)}_{12}, v^{(j)}_{11}\right\}$, *where*

$$w^{(j)}_{1t_{1_j}} = v^{(j)}_{1t_{1_j}} - \frac{q^* v^{(j)}_{11}}{1 + q^* v^{(1)}_{12}} \; v^{(1)}_{1,t_{1_j}+1}$$

$$w^{(j)}_{1l} = v^{(j)}_{1l} - \frac{q^* v^{(j)}_{11}}{1 + q^* v^{(1)}_{12}} \; v^{(1)}_{1,l+1} + \left(q^* v^{(j)}_{1,l+1} - \frac{q^* v^{(j)}_{11}}{1 + q^* v^{(1)}_{12}} \; q^* v^{(1)}_{1,l+2} \right) v_1$$

$$l = t_{1_j} - 1, t_{1_j} - 2, \ldots, 2, 1,$$

(2.2.2) $t_{1_1} \leq t_{1_j}$, *then the updated matrix* $A + v_1 q^*$ *has the following* k_1 *Jordan chains of length* $t_{1_1} - 1, t_{1_2} \ldots, t_{1_j} + 1, t_{1_{j+1}}, \ldots, t_{1_{k_1}}$,

(2.2.2.1) $\left\{w^{(1)}_{1,t_{1_1}-1}, w^{(1)}_{1t_{1_1}-2}, \ldots, w^{(1)}_{12}, w^{(1)}_{11}\right\}$, *of length* $t_{1_1} - 1$, *is the updated Jordan chain of* $\left\{v^{(1)}_{1t_{1_1}}, v^{(1)}_{1,t_{1_1}-1}, \ldots, v^{(1)}_{12}, v^{(1)}_{11} = v_1\right\}$, *where*

$$w^{(1)}_{1,t_{1_1}-1} = v^{(1)}_{1t_{1_1}} - \frac{1+q^*v^{(1)}_{12}}{q^*v^{(j)}_{11}}\ v^{(j)}_{1,t_1-1}$$

$$w^{(1)}_{1l} = v^{(1)}_{1,l+1} - \frac{1+q^*v^{(1)}_{12}}{q^*v^{(j)}_{11}}\ v^{(j)}_{1l}$$

$$+ \left(q^*v^{(1)}_{1,j+2} - \frac{1+q^*v^{(1)}_{12}}{q^*v^{(j)}_{11}}\ q^*v^{(j)}_{1,l+1} \right) v_1$$

$$l = t_{1_j} - 2, t_{1_j} - 3, \ldots, 2, 1,$$

(2.2.2.2) *For* $g = 2, 3, \ldots, k_1$, $g \neq j$, $\left\{ w^{(g)}_{1,t_{1_g}}, w^{(g)}_{1t_{1_g}}, \ldots, w^{(g)}_{12}, w^{(g)}_{11} \right\}$ *is the updated Jordan chain of* $\left\{ v^{(g)}_{1t_{1_g}}, v^{(g)}_{1,t_{1_g}-1}, \ldots, v^{(g)}_{12}, v^{(g)}_{11} \right\}$ *and it is defined as in case (1.2).*

(2.2.2.3) $\left\{ w^{(j)}_{1,t_{1_j}+1}, w^{(j)}_{1t_{1_j}}, \ldots, w^{(j)}_{12}, w^{(j)}_{11} \right\}$*, of length* $t_{1_j} + 1$*, is the updated Jordan chain of* $\left\{ v^{(j)}_{1t_{1_j}}, v^{(j)}_{1,t_{1_j}-1}, \ldots, v^{(j)}_{12}, v^{(j)}_{11} \right\}$*, where*

$$w^{(j)}_{1,t_{1_j}+1} = \frac{1}{q^*v^{(j)}_{11}}\ v^{(j)}_{1t_{1_j}}$$

$$w^{(j)}_{1l} = \frac{1}{q^*v^{(j)}_{11}}\ v^{(j)}_{1,l-1} + \frac{q^*v^{(j)}_{1l}}{q^*v^{(j)}_{11}}\ v_1, \qquad l = t_{1_j}, \ldots, 3, 2,$$

$$w^{(j)}_{11} = v_1.$$

(3) If there exist indices $j_1, j_2, \ldots, j_p$*, such that* $t_{1_{j_1}} \leq t_{1_{j_2}} \leq \cdots \leq t_{1_{j_p}}$*, with* $2 \leq j_i \leq k_1$ *and* $q^*v^{(j_i)}_{11} \neq 0$*,* $i = 1, 2, \ldots, p$*, we have:*

(3.1) If $t_{1_1} > t_{1_{j_p}}$ *then the updated matrix* $A + v_1 q^*$ *has the following* k_1 *Jordan chains of length* $t_{1_1}, t_{1_2} \ldots, t_{1_{k_1}}$*,*

(3.1.1) $\left\{ w^{(1)}_{1t_{1_1}}, w^{(1)}_{1,t_{1_1}-1}, \ldots, w^{(1)}_{12}, w^{(1)}_{11} \right\}$ *of length* t_{1_1}*, is the updated Jordan chain of* $\left\{ v^{(1)}_{1t_{1_1}}, v^{(1)}_{1,t_{1_1}-1}, \ldots, v^{(1)}_{12}, v^{(1)}_{11} = v_1 \right\}$ *and it is defined as in case (1.1).*

(3.1.2) For $2 \leq g \leq k_1$*,* $g \notin \{j_1, j_2, \ldots, j_p\}$*,* $\left\{ w^{(g)}_{1,t_{1_g}}, w^{(g)}_{1t_{1_g}-1}, \ldots, w^{(g)}_{12}, w^{(g)}_{11} \right\}$ *is the updated Jordan chain of* $\left\{ v^{(g)}_{1t_{1_g}}, v^{(g)}_{1,t_{1_g}-1}, \ldots, v^{(g)}_{12}, v^{(g)}_{11} \right\}$ *and it is defined as in case (1.2).*

(3.1.3) For $i = 1, 2, \ldots, p$*,* $\left\{ w^{(j_i)}_{1,t_{1_{j_i}}}, w^{(j_i)}_{1t_{1_{j_i}}-1}, \ldots, w^{(j_i)}_{12}, w^{(j_i)}_{11} \right\}$ *is the updated Jordan chain of* $\left\{ v^{(j_i)}_{1t_{1_{j_i}}}, v^{(j_i)}_{1,t_{1_{j_i}}-1}, \ldots, v^{(j_i)}_{12}, v^{(j_i)}_{11} \right\}$ *and it is defined as in case (2.2.1.3).*

(3.2) If $1 < t_{1_1} \leq t_{1_{j_p}}$ *then the updated matrix* $A + v_1 q^*$ *has the following* k_1 *Jordan chains of length* $t_{1_1} - 1, t_{1_2}, \ldots, t_{1_{j-1}}, t_{1_j} + 1, \ldots, t_{1_{k_1}}$*,*

(3.2.1) $\left\{w^{(1)}_{1,t_{1_1}-1}, w^{(1)}_{1t_{1_1}-2}, \ldots, w^{(1)}_{12}, w^{(1)}_{11}\right\}$, *of length* $t_{1_1}-1$, *is the updated Jordan chain of* $\left\{v^{(1)}_{1t_{1_1}}, v^{(1)}_{1,t_{1_1}-1}, \ldots, v^{(1)}_{12}, v^{(1)}_{11}=v_1\right\}$ *and defined as in case (2.2.2.1).*

(3.2.2) *For* $2 \le g \le k_1,\ g \notin \{j_1, j_2, \ldots, j_p\}$, $\left\{w^{(g)}_{1,t_{1_g}}, w^{(g)}_{1t_{1_g}}, \ldots, w^{(g)}_{12}, w^{(g)}_{11}\right\}$ *is the updated Jordan chain of* $\left\{v^{(g)}_{1t_{1_g}}, v^{(g)}_{1,t_{1_g}-1}, \ldots, v^{(g)}_{12}, v^{(g)}_{11}\right\}$ *and it is defined as in case (1.2).*

(3.2.3) $\left\{w^{(j)}_{1,t_{1_j}+1}, w^{(j)}_{1t_{1_j}}, \ldots, w^{(j)}_{12}, w^{(j)}_{11}\right\}$, *of length* $t_{1_j}+1$, *is the updated Jordan chain of* $\left\{v^{(j)}_{1t_{1_j}}, v^{(j)}_{1,t_{1_j}-1}, \ldots, v^{(j)}_{12}, v^{(j)}_{11}\right\}$ *and it is defined as in case (2.2.2.3).*

(3.2.4) *For* $i = 1, 2, \ldots, p-1$, $\left\{w^{(j_i)}_{1,t_{1_{j_i}}}, w^{(j_i)}_{1t_{1_{j_i}}-1}, \ldots, w^{(j_i)}_{12}, w^{(j_i)}_{11}\right\}$ *is the updated Jordan chain of* $\left\{v^{(j_i)}_{1t_{1_{j_i}}}, v^{(j_i)}_{1,t_{1_{j_i}}-1}, \ldots, v^{(j_i)}_{12}, v^{(j_i)}_{11}\right\}$, *where*

$$
\begin{aligned}
w^{(j_i)}_{1t_{1_{j_i}}} &= v^{(j_i)}_{1t_{1_{j_i}}} - \frac{q^* v^{(j_i)}_{11}}{q^* v^{(j_p)}_{11}} v^{(j_p)}_{1t_{1_{j_i}}} \\
w^{(j_i)}_{1l} &= v^{(j_i)}_{1l} - \frac{q^* v^{(j_i)}_{11}}{q^* v^{(j_p)}_{11}} v^{(j_p)}_{1l} \\
&\quad + \left(q^* v^{(j_i)}_{1,l+1} - \frac{q^* v^{(j_i)}_{11}}{q^* v^{(j_p)}_{11}}\, q^* v^{(j_p)}_{1,l+1} \right) v_1 \\
l &= t_{1_{j_i}} - 1, t_{1_{j_i}} - 2, \ldots, 2, 1.
\end{aligned}
$$

(3.3) *If* $t_1 = 1$ *then the updated matrix* $A + v_1 q^*$ *has* $k_1 - 1$ *Jordan chains of length* $t_{1_2}, t_{1_3}, \ldots, t_{1_{j-1}}, t_{1_j}+1, \ldots, t_{1_{k_1}}$ *defined as in cases (3.2.2), (3.2.3) and (3.2.4).*

Proof. It follows with the same techniques of the previous theorems. □

Example 3. Consider the matrix $A = J_3(\lambda_1) \oplus J_3(\lambda_1) \oplus J_2(\lambda_1)$ whose eigenvalue is λ_1, with 3 associated Jordan chains ($k_1 = 3$) of length $t_{1_1} = 3$, $t_{1_2} = 3$ and $t_{1_3} = 2$. These chains are

$$
\left\{v^{(1)}_{13}, v^{(1)}_{12}, v^{(1)}_{11}\right\} = \{e_3,\, e_2,\, e_1\}, \quad \left\{v^{(2)}_{13}, v^{(2)}_{12}, v^{(2)}_{11}\right\} = \{e_6,\, e_5,\, e_4\},
$$
$$
\left\{v^{(3)}_{12}, v^{(3)}_{11}\right\} = \{e_8,\, e_7\}
$$

where e_i denotes the ith unit vector, $i = 1, 2\ldots, 8$. We suppose that $v^{(1)}_{11} = e_1$ and $q^* = [q_1\ q_2\ q_3\ q_4\ q_5\ q_6\ q_7\ q_8]$. If $q^* e_1 = 0$ the matrix $A + v_1 q^*$ has the only eigenvalue λ_1. The generalized eigenvectors and the length of the Jordan chains depend on the values q_i, $i = 2, 3, \ldots, 8$. For example, if $q^* v^{(j)}_{11} = 0$, for $j = 2, 3$, we have that $q_4 = q_7 = 0$. We are in case (1), then the number of chains and their lengths do not change. The updated Jordan chains of $A + v_1 q^*$ are:

Jordan chains of A	Updated Jordan chains of $A + e_1 q^*$
$\{e_3, e_2, e_1\}$	$\left\{\frac{1}{1+q_2} e_3, \frac{1}{1+q_2} e_2 + \frac{q_3}{1+q_2} e_1, e_1\right\}$
$\{e_6, e_5, e_4\}$	$\{e_6, e_5 + q_6 e_1, e_4 + q_5 e_1\}$
$\{e_8, e_7\}$	$\{e_8, e_7 + q_8 e_1\}$

Note that in this case $A + v_1 q^*$ is similar to A.

4. Updating with a generalized eigenvector

In this section we give results when we replace an eigenvector of a matrix A by a generalized eigenvector of A in the rank one perturbation matrix. The following theorem can be considered as an extension of Brauer's Theorem [3] to a generalized eigenvector associated with the corresponding eigenvalue of A. For simplicity in the proof we consider the eigenvalue λ_1.

Theorem 5. *Let $A \in \mathbb{C}^{n\times n}$ with eigenvalues $\lambda_1, \lambda_2, \dots, \lambda_s$, and algebraic multiplicities $r_1, r_2, \dots, r_s$, respectively. Let $\{v_{1t_1}, v_{1,t_1-1}, \dots, v_{12}, v_{11}\}$ be a Jordan chain of A of length t_1, associated with λ_1. Let q be an n-dimensional vector such that $q^* v_{1h} = 0$, $h = 1, 2, \dots, k-1$, with $k \le t_1$. Then the matrix $A + v_{1k} q^*$ has eigenvalues $\lambda_1 + q^* v_{1k}, \lambda_1, \lambda_2, \dots, \lambda_s$ with algebraic multiplicities*

1. *$r_1, r_2, \dots, r_s$, if $q^* v_{1k} = 0$.*
2. *$1, r_1 - 1, r_2, \dots, r_s$, if $q^* v_{1k} \neq 0$ and $\lambda_1 + q^* v_{1k} \notin \sigma(A)$.*
3. *$r_1 - 1, r_2, \dots, r_{i-1}, r_i + 1, r_{i+1}, \dots, r_s$, if $q^* v_{1k} \neq 0$, $\lambda_1 + q^* v_{1k} \in \sigma(A)$ and $\lambda_1 + q^* v_{1k} = \lambda_i$, $1 < i \le s$.*

Proof. The idea of the proof is similar to that of Brauer's Theorem given in [9]. Consider the matrix $V_1 = [v_{11}\ v_{12}\ \dots\ v_{1,k-1}\ v_{1k}\ v_{1,k+1}\ \dots\ v_{1t_1}]$. Let $V = [V_1\ V_2]$ be a nonsingular matrix and $V^{-1} = \begin{bmatrix} U_1 \\ U_2 \end{bmatrix}$, such that

$$V^{-1} A V = \left[\begin{array}{c|c} J_{t_1}(\lambda_1) & U_1 A V_2 \\ \hline & U_2 A V_2 \end{array}\right],$$

where $\sigma(U_2 A V_2) = \{\lambda_1, \lambda_2, \dots, \lambda_s\}$, with algebraic multiplicities $r_1 - t_1, r_2, \dots, r_s$, respectively. Then

$$V^{-1}\left(A+v_{1k}q^*\right)V = V^{-1}AV + V^{-1}(v_{1k}q^*)V$$

$$= \left[\begin{array}{cccccc|c} J_{k-1}(\lambda_1) & E_1 & & & & & \\ & \lambda_1 + q^*v_{1k} & 1+q^*v_{1,k+1} & \cdots & q^*v_{1,t_1-1} & q^*v_{1t_1} & \\ & & \lambda_1 & & & & \\ & & & \ddots & \vdots & \vdots & U_1AV_2 + q^*V_2 \\ & & & & \lambda_1 & 1 & \\ & & & & & \lambda_1 & \\ \hline & & & & & & U_2AV_2 \end{array}\right],$$

where

$$E_1^T = [0\ 0\ \ldots\ 1]_{1\times(k-1)}.$$

Therefore

$$\sigma(A+v_{1k}q^*) = \{\lambda_1 + q^*v_{1k}, \lambda_1\} \cup \sigma(U_2AV_2),$$

and the result follows. □

Now, we study the relationship among the Jordan chains of a matrix A and the rank one updated matrix $A+v_{i_l}^{(j)}q^*$, where $v_{i_l}^{(j)}$ is a generalized eigenvector of A associated with λ_i, with the additional condition $q^*v_{i_h}^{(j)} = 0$, for $h = 1, 2, \ldots, l$ (note that in this case $\sigma(A) = \sigma(A+v_{i_l}^{(j)}q^*)$). Suppose that A has the eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_s$, with algebraic multiplicities $r_1, r_2, \ldots, r_s$, respectively, and for each eigenvalue λ_i, $i = 1, 2, \ldots, s$, there exist k_i Jordan chains of length $t_{i_1}, t_{i_2}, \ldots, t_{i_{k_i}}$. Then A has the Jordan structure

$$J_A = J(\lambda_1) \oplus J(\lambda_2) \oplus \cdots \oplus J(\lambda_s)$$

where

$$J(\lambda_i) = J_{t_{i_1}}(\lambda_i) \oplus J_{t_{i_2}}(\lambda_i) \oplus \cdots J_{t_{i_{k_i}}}(\lambda_i).$$

For $j = 1, 2, \ldots, k_i$, we denote by $\left\{v_{it_{i_j}}^{(j)}, v_{i,t_{i_j}-1}^{(j)}, \ldots, v_{i2}^{(j)}, v_{i1}^{(j)}\right\}$ the jth right Jordan chain associated with λ_i of length t_{i_j}, where

$$v_{ih}^{(j)} \in \left(\ker(A-\lambda_i I)^h\right) \setminus \left(\ker(A-\lambda_i I)^{h-1}\right), \qquad h = 1, 2, \ldots, t_{i_j},$$

and by $\left\{l_{i1}^{(j)^*}, l_{i2}^{(j)^*}, \ldots, l_{i,t_{i_j}-1}^{(j)^*}, l_{it_{i_j}}^{(j)^*}\right\}$ the jth left Jordan chain associated with λ_i of length t_{i_j}, where

$$l_{ih}^{(j)} \in \left(\ker(A^* - \overline{\lambda_i} I)^{t_{i_j}+1-h}\right) \setminus \left(\ker(A^* - \overline{\lambda_i} I)^{t_{i_j}-h}\right) \qquad h = 1, 2, \ldots, t_{i_j}.$$

The next results study how the Jordan chains of the rank one updated matrix $A + v_{i_l}^{(j)} q^*$ have changed from the Jordan chains of A. Different possibilities can be obtained in terms of the election of the generalized eigenvector $v_{i_l}^{(j)}$ and the vector q. Concretely, in Theorem 6 from two Jordan chains of length p and q of A associated with the eigenvalue λ_i we construct only one Jordan chain of $A + v_{i_l}^{(j)} q^*$ of length $p+q$ associated with the same eigenvalue. Theorem 7 obtains, for λ_i, two Jordan chains of length p and q of $A + v_{i_l}^{(j)} q^*$ from only one Jordan chain of length $p+q$ of A. Finally, in Theorem 8 we construct two Jordan chains of length r and s of $A + v_{i_l}^{(j)} q^*$ from two Jordan chains of length p and q of A, such that, $r + s = p + q$.

Theorem 6. *Let $A \in \mathbb{C}^{n\times n}$ with eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_s$, and algebraic multiplicities $r_1, r_2, \ldots, r_s$, respectively. Suppose that for each eigenvalue λ_i, $i = 1, 2, \ldots, s$, the matrix A has k_i Jordan chains of length $t_{i_1}, t_{i_2}, \ldots, t_{i_{k_i}}$. Let $q = l_{i1}^{(j)} \in \left(\ker(A^* - \overline{\lambda_i} I)^{t_{i_j}}\right) \setminus \left(\ker(A^* - \overline{\lambda_i} I)^{t_{i_j}-1}\right)$ be a left generalized eigenvector of the jth Jordan chain associated with the eigenvalue λ_i and let $v_{it_{j-1}}^{(j-1)} \in \left(\ker(A - \lambda_i I)^{t_{i_{j-1}}}\right) \setminus \left(\ker(A - \lambda_i I)^{t_{i_{j-1}}-1}\right)$ be a right generalized eigenvector of the $(j-1)$th Jordan chain associated with the same eigenvalue.*

Then the matrix $A + v_{it_{j-1}}^{(j-1)} q^*$ has, associated with λ_i, $k_i - 1$ Jordan chains of length $t_{i_1}, t_{i_2}, \ldots, t_{i_{j-2}}, t_{i_{j-1}} + t_{i_j}, t_{i_{j+2}}, \ldots, t_{i_{k_i}}$. Moreover, the Jordan structure of A and $A + v_{it_{j-1}}^{(j-1)} q^*$, associated with the eigenvalues λ_q, $1 \le q \le s$, $q \ne i$, is the same.

Proof. Since

$$q^* v_{ih}^{(j-1)} = l_{i1}^{(j)^*} v_{ih}^{(j-1)} = 0, \quad h = 1, 2, \ldots, t_{j-1},$$

the matrix $A + v_{it_{j-1}}^{(j-1)} q^*$ has the eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_s$, and algebraic multiplicities $r_1, r_2, \ldots, r_s$, respectively, by Theorem 5.

Now, to study the Jordan structure of this matrix consider the nonsingular matrix V such that

$$V^{-1} A V = J_A = J(\lambda_1) \oplus J(\lambda_2) \oplus \cdots \oplus J(\lambda_i) \oplus \cdots \oplus J(\lambda_{s-1}) \oplus J(\lambda_s)$$

and let g be the integer number

$$g = \sum_{l=1}^{i-1} \sum_{h=1}^{k_l} t_{l_h} + \left(t_{i_1} + t_{i_2} + \cdots + t_{i_{j-1}}\right).$$

Since

$$V^{-1}\left(A + v_{it_{j-1}}^{(j-1)} q^*\right) V = V^{-1} A V + \left(V^{-1} v_{it_{j-1}}^{(j-1)}\right)\left(l_{i1}^{(j)^*} V\right) = V^{-1} A V + e_g e_{g+1}^T,$$

where e_i is the ith unit vector, then

$$V^{-1}AV + e_g e_{g+1}^T = J_{A+v_{it_{j-1}}^{(j-1)}q^*} = J(\lambda_1) \oplus J(\lambda_2) \oplus \cdots \oplus \tilde{J}(\lambda_i) \oplus \cdots \oplus J(\lambda_{s-1}) \oplus J(\lambda_s)$$

with

$$\tilde{J}_{\lambda_i} = \begin{bmatrix} J_{t_{i_1}}(\lambda_i) & \dots & O & O & \dots & O \\ \vdots & & \vdots & \vdots & & \vdots \\ O & \dots & J_{t_{i_{j-1}}}(\lambda_i) & E & \dots & O \\ O & \dots & O & J_{t_{i_j}}(\lambda_i) & \dots & O \\ \vdots & & \vdots & \vdots & & \vdots \\ O & \dots & O & O & \dots & J_{t_{i_{k_i}}}(\lambda_i) \end{bmatrix},$$

and

$$E = \begin{bmatrix} 0 & 0 & \dots & 0 & 0 \\ 0 & 0 & \dots & 0 & 0 \\ \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & \dots & 0 & 0 \\ 1 & 0 & \dots & 0 & 0 \end{bmatrix}_{t_{i_{j-1}} \times t_{i_j}} . \qquad \square$$

Example 4. Consider a matrix A with eigenvalues λ_1 and λ_2, such that its the Jordan form is

$$J_A = V^{-1}AV = J_3(\lambda_1) \oplus J_2(\lambda_1) \oplus J_2(\lambda_2)$$

where

$$V = [v_{11}^{(1)}\ v_{12}^{(1)}\ v_{13}^{(1)}\ v_{11}^{(2)}\ v_{12}^{(2)}\ v_{21}^{(1)}\ v_{22}^{(1)}]$$
$$\left(V^{-1}\right)^* = \left[l_{11}^{(1)}\ l_{12}^{(1)}\ l_{13}^{(1)}\ l_{11}^{(2)}\ l_{12}^{(2)}\ l_{21}^{(1)}\ l_{22}^{(1)}\right]$$

are the matrices of the right and left generalized eigenvectors of A, respectively.

Taking $q^* = l_{11}^{(2)^*}$ and the vector $v_{13}^{(1)}$, the updated matrix $A + v_{13}^{(1)} l_{11}^{(2)^*}$ has the Jordan structure

$$V^{-1}\left(A + v_{13}^{(1)} l_{11}^{(2)^*}\right)V = V^1AV + V^{-1}\left(v_{13}^{(1)} l_{11}^{(2)^*}\right)V = J_5(\lambda_1) + J_2(\lambda_2).$$

Theorem 7. *Let $A \in \mathbb{C}^{n\times n}$ with eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_s$, and algebraic multiplicities $r_1, r_2, \ldots, r_s$, respectively. Suppose that for each eigenvalue λ_i, $i = 1, 2, \ldots, s$, the matrix A has k_i Jordan chains of length $t_{i_1}, t_{i_2}, \ldots, t_{i_{k_i}}$. Let $q = -l_{ip}^{(j)} \in \left(\ker(A^* - \overline{\lambda_i}I)^{t_{i_j}+1-p}\right) \setminus \left(\ker(A^* - \overline{\lambda_i}I)^{t_{i_j}-p}\right)$, $1 < p \le t_{i_j}$, be a left generalized eigenvector of the jth Jordan chain associated with the eigenvalue λ_i and let $v_{ip-1}^{(j)} \in$*

$\left(\ker(A-\lambda_i I)^{p-1}\right) \setminus \left(\ker(A-\lambda_i I)^{p-2}\right)$ be a right generalized eigenvector of the jth Jordan chain associated with the same eigenvalue.

Then the matrix $A+v_{ip-1}^{(j)}q^*$ has, associated with λ_i, k_i+1 Jordan chains of length $t_{i_1}, t_{i_2}, \ldots, t_{i_{j-2}}, t_{i_{j-1}}, t_{i_j}-p+1$, $p-1, t_{i_{j+1}}, \ldots, t_{i_{k_i}}$. Moreover, the Jordan structure of A and $A+v_{ip-1}^{(j)}q^*$, associated with the eigenvalues λ_q, $1 \le q \le s$, $q \ne i$, is the same.

Proof. The proof is analogous to that of Theorem 6. □

Example 5. Consider the matrix A given in Example 4. If we take the vector $q^* = -l_{12}^{(1)^*}$ the updated matrix $A+v_{11}^{(1)}(-l_{12}^{(1)^*})$ has the Jordan structure

$$\begin{aligned} V^{-1}\left(A+v_{11}^{(1)}(-l_{12}^{(1)^*})\right)V &= V^1AV + V^{-1}\left(v_{11}^{(1)}(-l_{12}^{(1)^*})\right)V \\ &= J_1(\lambda_1)\oplus J_2(\lambda_1)\oplus J_2(\lambda_1)\oplus J_2(\lambda_2). \end{aligned}$$

Theorem 8. *Let $A \in \mathbb{C}^{n\times n}$ with eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_s$, and algebraic multiplicities $r_1, r_2, \ldots, r_s$, respectively. Suppose that for each eigenvalue λ_i, $i=1,2,\ldots,s$, the matrix A has k_i Jordan chains of length $t_{i_1}, t_{i_2}, \ldots, t_{i_{k_i}}$. Let $q = l_{ip}^{(j)} \in \left(\ker(A^*-\overline{\lambda_i}I)^{t_{i_j}+1-p}\right) \setminus \left(\ker(A^*-\overline{\lambda_i}I)^{t_{i_j}-p}\right)$, $1 < p \le t_{i_j}$, be a left generalized eigenvector of the jth Jordan chain associated with the eigenvalue λ_i and let $v_{it_{j-1}}^{(j-1)} \in (\ker(A-\lambda_i I)^{t_{j-1}}) \setminus \left(\ker(A-\lambda_i I)^{t_{j-1}-1}\right)$ be a right generalized eigenvector of the $(j-1)$th Jordan chain associated with the same eigenvalue.*

Then the matrix $A+v_{it_{j-1}}^{(j-1)}q^*$ has, associated with λ_i, k_i Jordan chains of length $t_{i_1}, t_{i_2}, \ldots, t_{i_{j-2}}$, $t_{i_{j-1}}+t_{i_j}-p+1, p-1, t_{i_{j+2}}, \ldots, t_{i_{k_i}}$. Moreover, the Jordan structure of A and $A+v_{it_{j-1}}^{(j-1)}q^*$, associated with the eigenvalues λ_q, $1 \le q \le s$, $q \ne i$, is the same.

Proof. The proof is analogous to that of Theorem 6. □

Example 6. Again, consider the matrix A in Example 4. Now, taking $q^* = l_{12}^{(2)^*}$ and the vector $v_{13}^{(1)}$, we have

$$V^{-1}\left(A+v_{13}^{(1)}l_{12}^{(2)^*}\right)V = \left[\begin{array}{ccccc|cc} \lambda_1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & \lambda_1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & \lambda_1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & \lambda_1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & \lambda_1 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 0 & \lambda_2 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & \lambda_2 \end{array}\right],$$

which is similar to

$$J = J_4(\lambda_1)\oplus J_1(\lambda_1)\oplus J_2(\lambda_2).$$

Remark 1. We can obtain the same result as in Example 6 in two steps. First obtaining a Jordan chain of length 1 as in Example 5 and then applying Theorem 6 to matrix $A + v_{11}^{(1)}(-l_{12}^{(1)^*})$ to obtain a Jordan chain of length 4. That is

$$\begin{aligned} &V^{-1}\left(A + v_{11}^{(1)}(-l_{12}^{(1)^*}) + v_{13}^{(1)} l_{11}^{(2)^*}\right) V \\ &= V^{-1}\left(A + v_{11}^{(1)}(-l_{12}^{(1)^*})\right) V + V^{-1}\left(v_{13}^{(1)} l_{11}^{(2)^*}\right) V \\ &= J_1(\lambda_1) \oplus J_4(\lambda_1) \oplus J_2(\lambda_2) = V^{-1}\left(A + [v_{11}^{(1)}\ v_{13}^{(1)}] \begin{bmatrix} -l_{12}^{(1)^*} \\ l_{11}^{(2)^*} \end{bmatrix}\right) V. \end{aligned}$$

Note that in this case we apply a rank two perturbation.

5. Conclusions and open problems

We have studied the Jordan structure of the one rank updated matrix $A + v_k q^*$, where v_k is an eigenvector of A and q is an n-dimensional vector, in terms of the Jordan structure of A. Moreover, we have given the expressions of the generalized eigenvectors of the updated matrix. In particular, we have obtained the changes of the Jordan chains when the new eigenvalue $\mu \notin \sigma(A)$ in Section 2 and when $\mu \in \sigma(A)$ in Section 3. In both sections all Jordan chains associated with the eigenvalues of $A + v_k q^*$ are given.

When the updated matrix is constructed with a generalized eigenvector of A, instead of v_k, many different new possibilities appear as open problems. We have studied in Section 4 some of these possibilities when the spectrum of the updated matrix is the same as that of A.

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