Research Article

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Combined matrix of diagonally equipotent matrices

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Abstract: Let $C(A) = A \circ A^{-T}$ be the combined matrix of an invertible matrix A, where $\circ$ means the Hadamard product of matrices. In this work, we study the combined matrix of a nonsingular matrix, which is an H-matrix whose comparison matrix is singular. In particular, we focus on $C(A)$ when A is diagonally equipotent, and we study whether $C(A)$ is an H-matrix and to which class it belongs. Moreover, we give some properties on the diagonal dominance of these matrices and on their comparison matrices.

Keywords: H-matrix, combined matrix, irreducibility, diagonally equipotent matrix

MSC 2020: 15A15, 15A99, 15B99

1 Introduction

Let A be an $n \times n$ nonsingular matrix, and let $C(A)$ be its combined matrix (see [8]), that is,

$$C(A) = A \circ A^{-T},$$

where $\circ$ means the Hadamard product of matrices. The name of combined matrix appears for the first time in [8]. In control theory, it is known as "relative gain array" [4]. Moreover, the combined matrix has been applied in other subjects: (i) for constructing doubly diagonally stochastic matrices [10], (ii) to give a relationship between the eigenvalues and the diagonal entries [13], and (iii) to build a lower bound of the condition number [7].

Recall that the combined matrix of a reducible invertible matrix A is a block diagonal matrix whose diagonal blocks are the combined matrix of each irreducible diagonal block of the normal form of A (see [6]). Then, in this article, we always work with irreducible nonsingular matrices. We recall three basic properties of combined matrices:

$$C(A) = C(AD) = C(DA), \quad \text{where } D \text{ is nonsingular and diagonal.} \tag{1}$$

$$C(PAQ) = PC(A)Q, \quad \text{where } P \text{ and } Q \text{ are permutation matrices.} \tag{2}$$

$$\text{The entry sum of each row and column is 1.} \tag{3}$$

More properties and results of combined matrices can be seen in [1,6–11,13].

Various concepts of diagonal dominance of a matrix are essential in this work and are reviewed next.

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Definition 1. A matrix $A = [a_{ij}] \in \mathbb{C}^{n\times n}$ is said to be

(i) **generalized diagonally dominant** (GDD) if there exists an invertible nonnegative diagonal matrix D of size n such that AD is diagonally dominant (DD); that is,

$$|a_{ii}d_{ii}| \geq \sum_{j\neq i}|a_{ij}d_{ii}|, \quad i = 1, 2, \ldots, n.$$

(ii) **generalized strictly diagonally dominant** (GSDD) if all the aforementioned inequalities are strict.

(iii) **generalized diagonally equipotent** (GDE) if all the aforementioned inequalities are equalities.

Note that when the diagonal invertible matrix D is the identity matrix, the aforementioned three definitions become DD, strictly diagonally dominant (SDD), and diagonally equipotent (DE), respectively. Any DE matrix is denoted by DE as in [2]. In addition, we say that A is **irreducible DD** if the matrix is irreducible and at least one of the aforementioned inequalities is strict [17, Definition 1.7]. The rows have been considered in all those definitions.

Related to diagonal properties is the set of H-matrices (Section 2). We consider the three classes of H-matrices defined in [5]. The GSDD property is useful in the study of H-matrices of the invertible class ($\mathcal{H}_I$), that is, H-matrices whose comparison matrix is an invertible M-matrix.

We also work with H-matrices in the mixed class ($\mathcal{H}_M$) whose comparison matrix is a singular M-matrix, and there exists an invertible equimodular matrix. In both cases, we consider that all matrices are irreducible and so with nonnull diagonal elements since the class of a reducible H-matrix is determined by the properties of each square diagonal block [5, Theorems 3 and 5]. The class of singular H-matrices such that all equimodular matrices are singular is not considered.

In [6], the diagonally dominance of the combined matrix of H-matrices is studied. When the matrix is in the invertible class ($\mathcal{H}_I$), it follows that the combined matrix is an H-matrix in the same class $\mathcal{H}_I$ [6, Corollary 10] as it was given in [13, Lemma 5] and the combined matrix of an invertible matrix in $\mathcal{H}_M$ is an H-matrix [6, Corollary 19]. This last result does not differentiate between being in $\mathcal{H}_I$ or in $\mathcal{H}_M$ for the corresponding combined matrix.

In this work, we study the combined matrix of an invertible H-matrix in the mixed class ($\mathcal{H}_M$) shedding some light on the possible mentioned difference. In Section 2, we present some results on DE matrices. In Section 3, we study the combined matrix in some particular cases of these kinds of H-matrices denoted by DpM. In Section 4, we go one step further studying the combined of matrices in $\mathcal{H}_M$ denoted by DpMp1, that is, DpM matrices that have one more nonzero entry. In Section 5, we give our conclusions.

2 DE matrices

Let us see first that any irreducible and nonsingular H-matrix in $\mathcal{H}_M$ is DE. Recall that the *comparison matrix*, of an H-matrix A, $\mathcal{M}(A) = [m_{ij}]$ is defined as $m_{ii} = |a_{ii}|$ and $m_{ij} = -|a_{ij}|$ if $i \neq j$, for all $i, j = 1, 2, \ldots, n$.

First, we recall in Theorem 1 the relationship between the diagonally dominance property and general H-matrices. The first proposition can be seen in [5, Theorem 4]. The second part can be seen, for instance, in [6,14], where the results of the basic paper [16] were used.

Theorem 1. *Let A be an irreducible matrix. Then*

(*i*) *A is GDD if and only if A is an H-matrix.*

(*ii*) *A is GSDD if and only if A is in* $\mathcal{H}_I$.

Now, we have the following result for irreducible H-matrices in $\mathcal{H}_M$.

Theorem 2. *Let A be an irreducible matrix. The following conditions are equivalent*

(*i*) $A \in \mathcal{H}_M$.

(*ii*) *A is GDE.*

Proof. (i) ⇒ (ii). Since $A \in \mathcal{H}_M$, its comparison matrix $\mathcal{M}(A)$ is a singular M-matrix. Then there exists a positive vector d such that $\mathcal{M}(A)d = 0$ by [3, Theorem 6.4.16]. This property is essentially

$$\sum_j m_{ij} d_j = 0, \quad i = 1, 2, \ldots, n,$$

that is,

$$|a_{ii} d_i| = \sum_{j \neq i} |a_{ij} d_j|, \quad i = 1, 2, \ldots, n.$$

Taking $D = \mathrm{diag}(d_i)$, the aforementioned equation shows that the matrix AD is DE, and hence A is GDE.

(ii) ⇒ (i). The matrix A is an H-matrix since it is GDD by part (i) of Theorem 1. Moreover, the generalized diagonal dominance is not strict in any, row then A is not an $\mathcal{H}_I$ by part (ii) of Theorem 1. Then, the irreducible matrix A is in $\mathcal{H}_M$. □

It is worth saying that in the set of irreducible H-matrices, we have the following characterizations: $A \in \mathcal{H}_I \Leftrightarrow A$ is GSDD, and if A is irreducible $A \in \mathcal{H}_M \Leftrightarrow A$ is GDE. Moreover, if A is SDD, then A is $\in \mathcal{H}_I$ by [16, Theorem 1], and if A is irreducible DD, then A is $\in \mathcal{H}_I$ by [15, Theorem II].

We denote by M_{ij} the (i, j)-cofactor of $\mathcal{M}(A)$, that is, the (i, j)-minor with the corresponding sign $(-1)^{i+j}$.

Theorem 3. *Let A be an irreducible matrix, of order n, in $\mathcal{H}_M$. Let $\mathcal{M}(A)$ be its comparison matrix. Then, the cofactors M_{ij} of $\mathcal{M}(A)$ are constant in each row, that is,*

$$M_{ii} = M_{ij}, \quad \text{for all } i, j = 1, 2, \ldots, n.$$

Proof. Since A is in $\mathcal{H}_M$ then $\mathcal{M}(A)$ is singular. Then, by using the Lagrange expansion along the ith row,

$$\det(\mathcal{M}(A)) = -|a_{i1}| M_{i1} - |a_{i2}| M_{i2} - \cdots + |a_{ii}| M_{ii} - \cdots - |a_{in}| M_{in} = 0.$$

Then,

$$|a_{ii}| M_{ii} - \sum_{j \neq i} |a_{ij}| M_{ij} = 0,$$

and therefore,

$$\sum_{j \neq i} |a_{ij}| (M_{ii} - M_{ij}) = 0,$$

since $\mathcal{M}(A)$ is DE, that is, $a_{ii} = \sum_{j \neq i} |a_{ij}|$. Each term in the aforementioned sum is positive because $M_{ii} \geq M_{ij}$ by [6, Theorem 15]. Then

$$|a_{ij}| (M_{ii} - M_{ij}) = 0, \quad i, j = 1, 2, \ldots, n \quad \text{and} \quad i \neq j.$$

Case 1. If $a_{ij} \neq 0$, then $M_{ii} = M_{ij}$.

Case 2. If $a_{ik} = 0$, for some $k \neq i$, we construct the matrix $\overline{\mathcal{M}(B)} = [\overline{m_{ij}}]$ with the same entries as $\mathcal{M}(B)$, whose (i, i)th diagonal entry is $\overline{m_{ii}} = m_{ii} + 1$ and $\overline{m_{ik}} = -1$. The irreducible matrix $\overline{\mathcal{M}(B)}$ is a comparison matrix and so H-matrix, which is DE and then singular. Then, reasoning as before with this new matrix, we have

$$\overline{M_{ii}} = \overline{M_{ik}},$$

since $\overline{m_{ik}} \neq 0$. In addition, the cofactors of the ith row of $\mathcal{M}(B)$ are the same than those of $\overline{\mathcal{M}(B)}$. Then, $M_{ii} = M_{ik}$, even if $a_{ik} = 0$, for the given index i.

The aforementioned reasoning can be done for all rows, that is, for all $i = 1, 2, \ldots, n$. Hence, the cofactors of $\mathcal{M}(B)$ are constant in each row. □

3 Combined of DpM matrices

Let us start with the definition of DpM matrices. For that, we will work with monomial matrices. A square matrix A is monomial if $A = DP$, where P is a permutation matrix and D is an invertible diagonal matrix.

Definition 2. Let A be a matrix of order n. We say that A is **DpM** if can be written as the sum of a nonsingular **D**iagonal matrix **p**lus an irreducible **M**onomial matrix, that is,

$$A = D + M, \tag{4}$$

where D is a nonsingular diagonal matrix and M is an irreducible monomial matrix.

In the following steps, we prepare our initial matrix A using the two basic properties of combined matrices since we are going to study the combined matrix of a nonsingular (irreducible) DpM matrix in $\mathcal{H}_M$.

First step. We are working with matrix A in $\mathcal{H}_M$ which means that the matrix AD_1 is DE by Theorem 2, for some invertible nonnegative diagonal matrix D_1. The combined matrix of both matrices are equal by property (1). Then, we are working with a DE matrix.

Second step. With a convenient permutation matrix P, our matrix A can be written as follows:

$$\mathrm{PAP}^T = \begin{bmatrix} a_{11} & a_{12} & 0 & \cdots & 0 & 0 \\ 0 & a_{22} & a_{23} & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \ddots & & \vdots \\ \vdots & \vdots & & \ddots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & a_{n-1,n-1} & a_{n-1,n} \\ a_{n1} & 0 & 0 & \cdots & 0 & a_{nn} \end{bmatrix}.$$

Since $C(\mathrm{PAP}^T) = PC(A)P^T$ by property (2) of combined matrices, we assume that our invertible matrix in $\mathcal{H}_M$ has the above structure.

Third step. Finally, all diagonal elements are nonzero since our nonsingular H-matrix is irreducible by [5, Theorem 3]. We can multiply our matrix A on the right by the diagonal matrix $D_2 = \mathrm{diag}(1/a_{ii})$ and obtain the following matrix:

$$A = \begin{bmatrix} 1 & x_1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & x_2 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \ddots & & \vdots \\ \vdots & \vdots & & \ddots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 & x_{n-1} \\ x_n & 0 & 0 & \cdots & 0 & 1 \end{bmatrix}, \tag{5}$$

where $|x_i| = \frac{a_{i,i+1}}{a_{i+1,i+1}} = 1$, for all $i = 1, 2, \ldots, n$ with $a_{n,n+1} = a_{n,1}$ and $a_{n+1,n+1} = a_{11}$, since our matrix is DE. Note that $C(A) = C(AD_2)$ by Property 1. Accordingly, we suppose that our matrix in the following results of this section is the matrix A of (5).

Theorem 4. *Let A be a nonsingular DpM matrix in $\mathcal{H}_M$ of order n. Then, there exists a permutation matrix P such that*

$$PC(A)P^T = \begin{bmatrix} 1/2 & 1/2 & 0 & \cdots & 0 & 0 \\ 0 & 1/2 & 1/2 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \ddots & & \vdots \\ \vdots & \vdots & & \ddots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1/2 & 1/2 \\ 1/2 & 0 & 0 & \cdots & 0 & 1/2 \end{bmatrix}. \tag{6}$$

Proof. Suppose we have made all operations, of the last three steps, to our initial nonsingular irreducible DpM matrix in $\mathcal{H}_M$ to have the structure (5), included a symmetric permutation with the permutation matrix P.

The general expression of the determinant of a square matrix $M = [m_{ij}]$ is

$$\det M = \sum_{\sigma(i)\in\mathcal{P}_n} \mathrm{sgn}(\sigma) \prod_{i=n}^{n} m_{i,\sigma_i}, \tag{7}$$

where $\mathcal{P}_n$ is the set of all permutations σ of the n first natural numbers and σ_i denotes the position of the ith number after the permutation σ. The determinant of our matrix (5) is

$$\det A = 1 + \operatorname{sgn}\{2, 3, \ldots, n, 1\}\prod_{i=1}^{n} x_i,$$

where the symbol "sgn" stands for the sign of the permutation $\{2, 3, \ldots, n, 1\}$ of the n first natural numbers. We have

$$s = \operatorname{sgn}\{2, 3, \ldots, n, 1\} = \begin{cases} +1, & \text{if } n = 2k + 1, \\ -1, & \text{if } n = 2k, \end{cases}$$

for $k \in \mathbb{N}$. Then, there are two possibilities

$$\det A = \begin{cases} 2, & \text{if } s \quad \text{and} \quad t \text{ have the same sign}, \\ 0, & \text{if } s \quad \text{and} \quad t \text{ have different sign}, \end{cases}$$

where $t = \Pi_{i=1}^{n} x_i$. We work only in the nonsingular case and then $\det A = 2$.

The (1,1) entry of $C(A)$ is

$$c_{11} = \frac{a_{11}A_{11}}{\det A} = \frac{1}{2}\cdot$$

since the cofactor of the triangular block A_{11} is 1. Recalling that the combined matrix preserves the zero pattern of A and the sum of the entries of each row and column is 1 [12], we have

$$c_{12} = c_{n1} = \frac{1}{2}\cdot$$

Reasoning recursively with these two properties, we obtain that all nonnull entries of the combined matrix are exactly $\frac{1}{2}\cdot$ Consequently, the combined matrix of the initial DpM matrix in $\mathcal{H}_M$ is a symmetric permutation of (6). □

Theorem 5. *Let A be a nonsingular DpM matrix in $\mathcal{H}_M$ of order n. Then, $C(A)$ is an irreducible H-matrix in $\mathcal{H}_M$, which is nonsingular if $n = 2k + 1$ and singular when $n = 2k$, $k \in \mathbb{N}$.*

Proof. We assume that our matrix A is given by (5). From the general expression of the determinant of a square matrix $M = [m_{ij}]$ given in (7) and observing the structure of the matrix (6), we note that there are only two permutations

$$\{1, 2, \ldots, n\} \quad \text{and} \quad \{2, 3, \ldots, n, 1\},$$

where the product $\prod_{i=n}^{n} m_{i,\sigma_i}$ is not null. Since $\operatorname{sgn}\{1, 2, 3, \ldots, n\} = 1$, the determinant of the combined matrix $C(A)$ given in (6) is

$$\det(C(A)) = \left(\frac{1}{2}\right)^n + \operatorname{sgn}\{2, 3, \ldots, n, 1\}\left(\frac{1}{2}\right)^n \cdot$$

Then, the combined matrix is invertible if $n = 2k + 1$ since $\operatorname{sgn}\{2, 3, \ldots, n, 1\} = 1$, and the combined matrix is singular if $n = 2k, k \in \mathbb{N}$ as $\operatorname{sgn}\{2, 3, \ldots, n, 1\} = -1$. In addition, the combined matrix is irreducible and DE by (6). Hence, the irreducible matrix $C(A)$ is in $\mathcal{H}_M$ by Theorem 2. □

Note that the cofactors of the companion matrix of $C(A)$, that is, $\mathcal{M}(C(A))$ are equal in each row by Theorem 3. For instance, if we suppose that the matrix (6) has order 5, then the cofactors of row one are equal to 0.0625.

Define the map $\Phi(A) = C(A)$. In [13], the limit

$$\lim_{k\to\infty} \Phi_k(A)$$

is studied for H-matrices with nonsingular comparison matrix, that is, matrices in the invertible class.

Theorem 6. [13, Theorem 2]. *If A is an H-matrix in $\mathcal{H}_I$, then* $\lim_{k\to\infty}\Phi_k(A) = I$.

In our case, that is, when we are working with nonsingular H-matrices in $\mathcal{H}_M$, we have different results.

Theorem 7. *Let A be a nonsingular DpM matrix in $\mathcal{H}_M$ of order n and consider* $\lim_{k\to\infty}\Phi_k(A)$. *Then*

(*i*) *if* $n = 2k + 1$

$$\lim_{k\to\infty} \Phi_k(A) = C(A),$$

where $C(A)$ is a fixed point of $\Phi(\cdot)$.

(*ii*) *If* $n = 2k$, $\lim_{k\to\infty}\Phi_k(A)$ *does not exist.*

Proof. The combined matrix of A is a symmetric permutation of (6) by Theorem 4.

(i) If $n = 2k + 1$, then $C(A)$ given by (6) is invertible by Theorem 4. It is clear that the invertible $C(A)$ is a DpM matrix in $\mathcal{H}_M$ and can be considered as (5) using the diagonal matrix $D_2 = \operatorname{diag}\left(\frac{1}{2}\right)$. By applying Theorem 5, we have

$$\Phi(C(A)) = C(A),$$

that is, $C(A)$ is a fixed point of the map $\Phi(\cdot)$. Therefore,

$$\lim_{k\to\infty} \Phi_k(A) = C(A).$$

(ii) If the order of A is $n = 2k$, there is nothing to prove because $C(A)$ given in (6) is singular by Theorem 5. Therefore, the limit does not exist. □

4 Combined of DpMp1 matrices

Now, let us go one step further in the study of combined matrices of H-matrices in $\mathcal{H}_M$. For that, we are going to work with DpM matrices when we add exactly one nonzero entry.

Definition 3. Let A be a matrix of order n. We say that A is **DpMp1** if is DpM with exactly one more nonzero entry in any place of the zero pattern of A.

Note that, a DpMp1 matrix can be written as the sum of a nonsingular **D**iagonal matrix **p**lus an irreducible **M**onomial matrix **p**lus **1** nonzero entry as follows:

$$A = D + M + E_{ij}, \tag{8}$$

where D is a nonsingular diagonal matrix, M is an irreducible monomial matrix, and E_{ij} is a matrix with exactly one nonzero entry in the (i, j)th position, $i \neq j$, not included in the nonzero pattern of M.

The first thing we note in this case is that despite our matrix A being irreducible, its combined matrix can be reducible as seen in the following example.

Example 1. The DpMp1 matrix

$$A = \begin{bmatrix} 1 & 1/2 & -1/2 \\ 0 & 1 & -1 \\ -1 & 0 & 1 \end{bmatrix},$$

is irreducible in $\mathcal{H}_M$. However, its combined matrix

$$C(A) = \begin{bmatrix} 1 & 1/2 & -1/2 \\ 0 & 1/2 & 1/2 \\ 0 & 0 & 1 \end{bmatrix},$$

is a triangular matrix and thereby reducible.

Again, some manipulations are needed in our DpMp1 invertible matrix in $\mathcal{H}_M$ to obtain our result.

First step. Our initial matrix A is in $\mathcal{H}_M$. It means that the matrix AD_1 is DE, for some nonnegative nonsingular diagonal matrix D_1. Since the combined matrix of both matrices are equal, that is, $C(A) = C(AD_1)$, we will work with a DE matrix in the proof.

Second step. With a convenient symmetric permutation our matrix can be written as follows:

$$\mathrm{PAP}^T = D + \mathrm{PMP}^T + PE_{ij}P^T,$$

with the structure

$$\begin{bmatrix} a_{11} & a_{12} & 0 & \cdots & 0 & a_{1n} \\ 0 & a_{22} & a_{23} & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \ddots & & \vdots \\ \vdots & \vdots & & \ddots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & a_{n-1,n-1} & a_{n-1,n} \\ a_{n1} & 0 & 0 & \cdots & 0 & a_{nn} \end{bmatrix}, \tag{9}$$

where we have denoted all nonzero entries by a_{ij} for simplicity, and we have chosen the new nonzero entry in the upper right corner, without loss of generality.

Third step. Finally, as all diagonal elements are nonnull since our nonsingular matrix in $\mathcal{H}_M$ is irreducible, we can multiply our matrix by the diagonal matrix $D_2 = \mathrm{diag}(1/a_{ii})$ and obtain the matrix

$$A = \begin{bmatrix} 1 & x & 0 & \cdots & 0 & y \\ 0 & 1 & x_2 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \ddots & & \vdots \\ \vdots & \vdots & & \ddots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 & x_{n-1} \\ x_n & 0 & 0 & \cdots & 0 & 1 \end{bmatrix}, \tag{10}$$

where $|x| = \frac{a_{12}}{a_{22}}$, $|y| = \frac{a_{1n}}{a_{nn}}$ and $|x_i| = \frac{a_{i,i+1}}{a_{i+1,i+1}} = 1$, for all $i = 2, \dots, n$, with $a_{n,n+1} = a_{n,1}$, $a_{n+1,n+1} = a_{11}$ and

$$|x| + |y| = 1, \quad x \neq 0 \neq y \quad \text{and} \quad |x_i| = 1, \quad i = 2, 3, \dots, n. \tag{11}$$

For now on, we suppose our matrix A will have the three operations made in the three steps and will work with the matrix (10), with the conditions (11). Matrices DpMp1 in $\mathcal{H}_M$ can have the following determinants.

Theorem 8. *Let A be a matrix DmMp1 in $\mathcal{H}_M$ of order n as the matrix* (10). *Then,*

$$\det A = 1 \pm |x| \pm |y|, \tag{12}$$

given four possibilities: (1) $\det A = 2|x|$, (2) $\det A = 2|y|$, (3) $\det A = 2$, *and* (4) $\det A = 0$.

Proof. Again, from the general expression of the determinant of a matrix (7) and observing the structure of the matrix (10), we note that there are only three permutations

$$\{1, 2, \dots, n\} \quad \{2, 3, \dots, n, 1\} \quad \text{and} \quad \{n, 2, 3, \dots, n-1, 1\},$$

where the corresponding product $\prod_{i=n}^{n} a_{i,\sigma_i}$ is not null. The expression (7) reduces to

$$\det A = \mathrm{sgn}\{1, 2, \dots, n\} \cdot 1 + \mathrm{sgn}\{2, 3, \dots, n, 1\} x \prod_{i=2}^{n} x_i + \mathrm{sgn}\{n, 2, 3, \dots, n-1, 1\} x_n y,$$

that is,

$$\det A = 1 \pm |x| \pm |y|,$$

since all signs, x_n and $\prod_{i=2}^{n} x_i$ are ± 1.

Let us see the four possible values of (12).

(1) If $\det A = 1 + |x| - |y|$, we have $\det A = 2|x|$ by equation (11).

(2) If $\det A = 1 - |x| + |y|$, then $\det A = 2|y|$ since $-|x| = |y| - 1$.

(3) If $\det A = 1 + |x| + |y|$, then $\det A = 2$ by equation (11).
(4) Finally, if $\det A = 1 - |x| - |y|$, we have a singular matrix since $\det A = 0$ by (11). □

Remark 1. Theorem 8 remains valid if the new nonzero entry y will appear in any other position of the upper triangular part of A since the corresponding entries x_i, x and y will satisfy conditions (11). Moreover, the same result will happen if the new nonzero entry is in the lower triangular part.

Theorem 9. *Let A be an invertible matrix DpMp1 in $\mathcal{H}_M$ of order n. Then,*
(*i*) *$C(A)$ is GDD with at least one row strictly diagonal dominant*;
(*ii*) *the combined matrix $C(A)$ is an H-matrix in $\mathcal{H}_I$.*

Proof. By properties (1–2) of a combined matrix, we can assume that we are working with a matrix as (10). We consider three cases according to the nonzero values of $\det A$ given in Theorem 8.

Case 1. $\det A = 2|x|$.

Proof of (i). The entries of $C(A)$ are as follows:

$$c_{ij} = \frac{1}{\det A} a_{ij} A_{ij},$$

where A_{ij} is the cofactor of a_{ij}.

Then, we have

$$c_{11} = \frac{1}{\det A},$$

since A_{11} is the determinant of a triangular submatrix with 1's in the diagonal. By property (3) of a combined matrix, we have

$$c_{n1} = 1 - c_{11} = 1 - \frac{1}{\det A} = \frac{|x| - |y|}{\det A},$$
$$c_{nn} = 1 - c_{n1} = \frac{1}{\det A}\cdot$$

Now, by using the Laplace expansion of the determinant of the matrix (10), by the cofactors of the first row, we have

$$\det A = a_{11}A_{11} + a_{12}A_{12} + a_{1n}A_{1n} = 1 + c_{12}\det A + c_{1n}\det A.$$

Moreover,

$$c_{12} = \frac{1}{\det A} a_{12}A_{12} = \frac{|x|}{\det A}\cdot$$

By comparing these last two expressions with the value of $\det A = 2|x| = 1 + |x| - |y|$ in this case, we conclude

$$c_{1n} = \frac{-|y|}{\det A}\cdot$$

By applying property (3) of combined matrices to the second column, we have $c_{22} = \frac{1-|y|}{\det A}\cdot$ Similarly, $c_{23} = \frac{|x|}{\det A}\cdot$ Working in this way, with the following rows and columns, we obtain

$$C(A) = \frac{1}{\det A}\begin{bmatrix} 1 & |x| & 0 & 0 & \cdots & 0 & -|y| \\ 0 & 1-|y| & |x| & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \ddots & & & \vdots \\ \vdots & \vdots & & \ddots & \ddots & & \vdots \\ \vdots & \vdots & & & \ddots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & 1-|y| & |x| \\ |x|-|y| & 0 & 0 & 0 & \cdots & 0 & 1 \end{bmatrix}. \tag{13}$$

Note that the last row satisfies the inequality

$$1 > |x| - |y|,$$

by (11). All other rows are DD by (11). Then, the matrix (13) is DD with one row SDD and so our matrix A will be GDD with one SDD row.

Proof of (ii). All entries of $C(A)$ of (13) defined in function of x or y are not null except, maybe the $(n, 1)$ entry. Here, we consider two possibilities.

(a) $|x| \neq |y|$. In this case, our combined matrix (13) is irreducible with positive diagonal entries, with the last row strict DD as well as $\mathcal{M}(C(A))$. Therefore, $\mathcal{M}(C(A))$ is invertible by [15, Theorem II]. Hence, $C(A)$ is an H-matrix in $\mathcal{H}_I$.

(b) $|x| = |y|$. In this case, the combined matrix (13) and $\mathcal{M}(C(A))$ are triangular with nonzero diagonal entries by (11). Then the comparison matrix is invertible, and therefore, $C(A)$ is an H-matrix in $\mathcal{H}_I$.

Case 2. $\det A = 2|y|$.

Proof of (i). Following the same work as Case 1, we obtain

$$C(A) = \frac{1}{\det A}\begin{bmatrix} 1 & -|x| & 0 & 0 & \cdots & 0 & |y| \\ 0 & 1+|y| & -|x| & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \ddots & & & \vdots \\ \vdots & \vdots & & \ddots & \ddots & & \vdots \\ \vdots & \vdots & & & \ddots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & 1+|y| & -|x| \\ -|x|+|y| & 0 & 0 & 0 & \cdots & 0 & 1 \end{bmatrix}. \tag{14}$$

In this case, the first row of (14) is DD due to (11). However, the last row is SDD because

$$|-|x| + |y|| < 1.$$

In addition, all inner rows of (14) are SDD since $|x| = 1 - |y|$, and then

$$|x| < 1 + |y|.$$

Then, $C(A)$ is a GDD matrix with $n - 1$ strictly diagonal dominant rows.

Proof of (ii). The only entry of the combined (14), defined in function of x and y, that can be zero is the $(n, 1)$. We have two cases.

(a) $|x| \neq |y|$. The comparison matrix $\mathcal{M}(C(A))$ of the combined matrix (14) is irreducible with positive diagonal entries and DD with $n - 1$ SDD rows. Therefore, $\mathcal{M}(C(A))$ is invertible by [15, Theorem II]. Hence, $C(A)$ is an H-matrix in $\mathcal{H}_I$.

(b) $|x| = |y|$. Now, the comparison matrix $\mathcal{M}(C(A))$ of the combined matrix (14) is triangular with nonzero diagonal entries by (11). Then, it is invertible and therefore $C(A)$ is an H-matrix in $\mathcal{H}_I$.

Case 3. $\det A = 2$.

Proof of (i). Following the proof of (i), in any of the two aforementioned cases, we obtain

$$C(A) = \frac{1}{\det A}\begin{bmatrix} 1 & |x| & 0 & 0 & \cdots & 0 & |y| \\ 0 & 1+|y| & |x| & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \ddots & & & \vdots \\ \vdots & \vdots & & \ddots & \ddots & & \vdots \\ \vdots & \vdots & & & \ddots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & 1+|y| & |x| \\ |x|+|y| & 0 & 0 & 0 & \cdots & 0 & 1 \end{bmatrix}. \tag{15}$$

Bearing in mind the equality (11), we see that the first and the last rows of (15) are DD and all inner rows are SDD. Then, $C(A)$ is a GDD matrix with $n - 2$ SDD rows.

Proof of (ii). In this case, the combined matrix (15) is always irreducible. Then, $\mathcal{M}(C(A))$ is irreducible DD with $n - 2$ SDD rows. Therefore, $\mathcal{M}(C(A))$ is invertible by [15, Theorem II]. Hence, $C(A)$ is in $\mathcal{H}_I$. □

The result of Theorem 6 can be used to compute the limit of the map Φ when A is a DpMp1 matrix in $\mathcal{H}_M$.

Theorem 10. *Let A be a nonsingular DpMp1 matrix in $\mathcal{H}_M$. Then,*

$$\lim_{k\to\infty} \Phi_k(A) = I.$$

Proof. We have that $\Phi(A) = \mathcal{C}(A)$ is an H-matrix in $\mathcal{H}_I$ by Theorem 9. Then

$$\lim_{k\to\infty} \Phi_k(A) = I,$$

by Theorem 6. □

5 Conclusion

In this article, we have studied the character of the combined matrix of an irreducible nonsingular H-matrix in $\mathcal{H}_M$. Bearing in mind that an irreducible H-matrix in $\mathcal{H}_M$ is GDE, two kinds of H-matrices in $\mathcal{H}_M$ have been analyzed. First, the special case of DpM matrices are considered showing that their corresponding combined matrix is always irreducible and GDE matrix in $\mathcal{H}_M$ given by a symmetric permutation of (6). The combined matrix can be singular or not, depending on whether the order of the matrix is even or odd, respectively. Second, the combined matrix of a DpMp1 matrix in $\mathcal{H}_M$ is studied. In particular, it turns out that the combined matrix of a DpMp1 matrix is always a nonsingular H-matrix in $\mathcal{H}_I$. However, in this case, the combined matrix can be reducible. The variability of the results of this work shows that different cases may occur in other matrices. This is our future work. In addition, the limit of the map $\Phi_k(A) = \mathcal{C}(A)$, when k goes to infinity, has been considered with those two matrices. When A is DpM, the limit may not exist if the order of the matrix is $n = 2k$ or is the combined matrix itself if $n = 2k + 1$, $k = 1, 2, \ldots$ When A is DpMp1, the limit of $\Phi_k(A)$, when k goes to infinity, is the identity matrix as in the case of H-matrices in $\mathcal{H}_I$.

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