# Three-mass and springs: modelling recap and normalised matrix form

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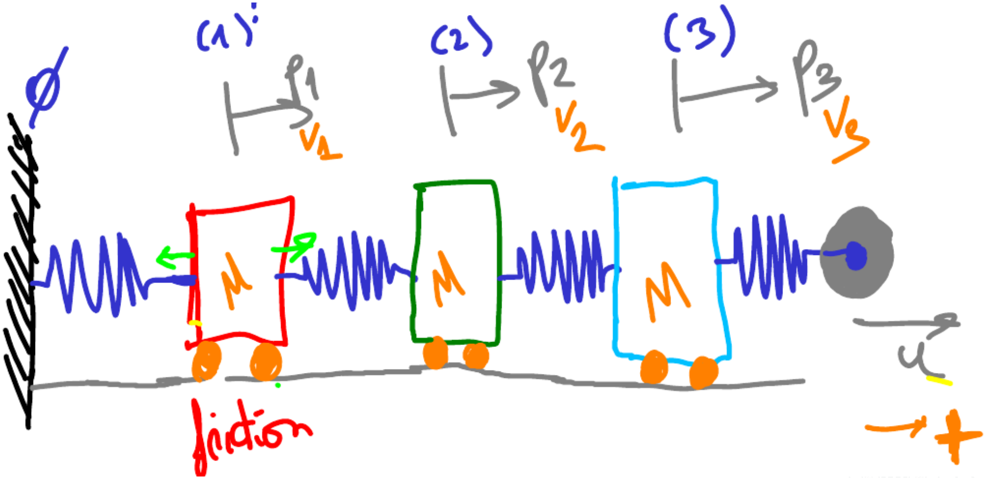
Systems Modelling

DISA – ETSII – Universitat Politècnica de València

Video-presentación disponible en:

http://personales.upv.es/asala/YT/V/**moll3mod2EN**.html

# Sketch of the physical system



# Model in "absolute" coordinates (affine, linear+const.)

Origin is left black wall (not moving), $l_{n\cdot}$ denotes "natural length".

$$\frac{dP_1}{dt} = v_1$$

$$\frac{dv_1}{dt} = \frac{1}{M_1}\left(-k_1(P_1 - l_{n1}) - bv_1 + k_2(P_2 - P_1 - l_{n2})\right)$$

$$\frac{dP_2}{dt} = v_2$$

$$\frac{dv_2}{dt} = \frac{1}{M_2}\left(-k_2(P_2 - P_1 - l_{n2}) + k_3(P_3 - P_2 - l_{n3}) - bv_2\right)$$

$$\frac{dP_3}{dt} = v_3$$

$$\frac{dv_3}{dt} = \frac{1}{M_3}\left(-k_3(P_3 - P_2 - l_{n3}) - bv_3 + k_4(U - P_3 - l_{n4})\right)$$

# Model in "absolute" coordinates: equilibrium

$$0 = v_1^{eq}$$

$$0 = \frac{1}{M_1}\left(-k_1(P_1^{eq} - l_{n1}) - bv_1^{eq} + k_2(P_2^{eq} - P_1^{eq} - l_{n2})\right)$$

$$0 = v_2^{eq}$$

$$0 = \frac{1}{M_2}\left(-k_2(P_2^{eq} - P_1^{eq} - l_{n2}) + k_3(P_3^{eq} - P_2^{eq} - l_{n3}) - bv_2^{eq}\right)$$

$$0 = v_3^{eq}$$

$$0 = \frac{1}{M_3}\left(-k_3(P_3^{eq} - P_2^{eq} - l_{n3}) - bv_3^{eq} + k_4(U^{eq} - P_3^{eq} - l_{n4})\right)$$

# Model in “INCREMENTAL” variables (linear)

Origin is the equilibrium point in prev. slide. Natural lengths disappear.

$$\frac{dp_1}{dt} = v_1$$

$$\frac{dv_1}{dt} = \frac{1}{M_1}\left(-(k_1+k_2)p_1 - bv_1 + k_2 p_2\right)$$

$$\frac{dp_2}{dt} = v_2$$

$$\frac{dv_2}{dt} = \frac{1}{M_2}\left(k_2 p_1 - (k_2+k_3)p_2 - bv_2 + k_3 p_3\right)$$

$$\frac{dp_3}{dt} = v_3$$

$$\frac{dv_3}{dt} = \frac{1}{M_3}\left(k_3 p_2 - (k_3+k_4)p_3 - bv_3 + k_4 u\right)$$

# Model in “INCREMENTAL” variables (linear)

We’ll set all masses to be equal, as well as springs, for later simulations.

$$\frac{dp_1}{dt} = v_1$$

$$\frac{dv_1}{dt} = \frac{1}{M}\left(-2kp_1 - bv_1 + kp_2\right)$$

$$\frac{dp_2}{dt} = v_2$$

$$\frac{dv_2}{dt} = \frac{1}{M}\left(kp_1 - 2kp_2 - bv_2 + kp_3\right)$$

$$\frac{dp_3}{dt} = v_3$$

$$\frac{dv_3}{dt} = \frac{1}{M}\left(kp_2 - 2kp_3 - bv_3 + ku\right)$$

# Normalised internal representation(state eq.)

The normalised representation $\frac{dx}{dt} = Ax + Bu$ is written as:

$$\overbrace{\frac{d}{dt}\begin{pmatrix} p_1 \\ v_1 \\ p_2 \\ v_2 \\ p_3 \\ v_3 \end{pmatrix}}^{\frac{dx}{dt}} = \overbrace{\begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ \frac{-2k}{M} & \frac{-b}{M} & \frac{k}{M} & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ \frac{k}{M} & 0 & \frac{-2k}{M} & \frac{-b}{M} & \frac{k}{M} & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & \frac{k}{M} & 0 & \frac{-2k}{M} & \frac{-b}{M} \end{pmatrix}}^{A} \overbrace{\begin{pmatrix} p_1 \\ v_1 \\ p_2 \\ v_2 \\ p_3 \\ v_3 \end{pmatrix}}^{x} + \overbrace{\begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ \frac{k}{M} \end{pmatrix}}^{B} u$$

*Output equation is arbitrary, depending on concrete applications.

For instance, $y = x$, i.e., $C = I$, $D = 0$ in $y = Cx + Du$ extracts the 6 positions and velocities.

Equilibrium point will always be $(x = 0, y = 0)$ if $u = 0$ (we are in incremental coordinates).