Dynamics of a non-rigid system of particles: moments (mean and variance, informally), 2D planar motion

Lecture notes for aerospace/robotics engineers

Antonio Sala, Universitat Politècnica de València

Videos:

http://personales.upv.es/asala/YT/V/**partic1EN**.html (center of mass, Euler mass matrix, dynamics in 1st and 2nd moments), http://personales.upv.es/asala/YT/V/**partic2EN**.html (least squares fit of a velocity field), http://personales.upv.es/asala/YT/V/**partic3EN**.html (angular momentum conservation, equations of rigid-body motion), http://personales.upv.es/asala/YT/V/**angmom2dEN**.html (interpretation of angular momentum)

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Outline

Motivation:

In an abstract conceptual framework, all mechanical systems are a set of "particles" (point masses) interacting.

Objectives:

In the same way we study random variables with mean and variance, we can do the same with an ensemble of such particles. We can characterize the "rigid" mode of the motion and the "deformation" one (quadratic approximation).

Contents:

Mean motion (center of mass), mean square (variance) motion: momentum, mass matrix, virials. Interpretation.

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A rigid (or deformable) body consisting of N particles (2D truss)

We have x_i, $i = 1, \ldots, N$, with $x_i \in \mathbb{R}^2$.
The equations of motion can be written as:

- Derivative of position is speed: $\frac{dx_i}{dt} = v_i$
- **Newton's second law:** $m_i \frac{dv_i}{dt} = f_i^{res} = f_i^{ext} + f_i^{int}$

Compact notation $X = [x_1\ x_2\ \ldots x_N]_{2\times N}$, $M = diag([m_1\ \ldots\ m_N])$, and likewise for F^{ext}, F^{int}.
Equations of motion of all particles:

$$\frac{dX_w}{dt}_{2\times N} = V_{2\times N}, \qquad \frac{dV}{dt}_{2\times N} \cdot M_{N\times N} = F^{ext}_{2\times N} + F^{int}_{2\times N}$$

Multiplying by M the left equation defines "linear momentum vector":
$\frac{dX_w}{dt}_{2\times N} M_{N\times N} = V_{2\times N} M_{N\times N} = P_{2\times N}$

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Internal and external forces

- **Exterior forces:** Exogenous applied forces (assumed known) plus reaction forces at some points with prefixed positions (unknowns).
- **Generalized third law (interior forces):** (interior) force between two particles is opposite (action-reaction) and **collinear**, $f_{ij}^{int} = \kappa_{ij}(x_i - x_j)$, $\kappa_{ij} = \kappa_{ji} \Rightarrow f_{ji}^{int} = -f_{ij}^{int}$. Also, we set $k_{ii} = 0$.

  Note: The above holds if (interior) forces come from a potential energy function depending on positions $V(x_{1,b}, x_{2,b}, \dots)$ that does not change with rotations or translations. Electromagnetism breaks it: angular momentum can be created from linearly moving charges (electric motor), as Lorentz forces depend on speed. But we are doing "XVIII-century mechanics" with sort of "spring-connected particles" or "gravity potentials".

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Motion analysis in "mean and variance" (moments) [1]

MEAN: center of mass (CM) motion

Consider a vector $\mathbf{1}_{N\times 1}$, all ones. Total mass $m = \mathbf{1}^T M_{N\times N}\mathbf{1} = \sum_i m_i$.
Define $x_{cm} = (X \cdot M \cdot \mathbf{1}_{N\times 1})/m$.
We have:

$$m\frac{dx_{cm}}{dt}_{2\times 1} = V_{2\times N}\cdot M_{N\times N}\cdot \mathbf{1}_{N\times 1} = P_{2\times N}\cdot \mathbf{1}_{N\times 1} = p_{cm,2\times 1}$$

i.e., "total linear momentum" is "total mass" times "center of mass velocity". Regarding acceleration:

$$m\frac{d^2x_{cm}}{dt} = \frac{dV}{dt}_{2\times N}\cdot M_{N\times N}\mathbf{1}_{N\times 1} = F^{ext}_{2\times N}\mathbf{1}_{N\times 1} + F^{int}_{2\times N}\mathbf{1}_{N\times 1} = F^{ext}_{2\times N}\mathbf{1}_{N\times 1}$$

*Recall that by 3rd law, internal forces fulfill $F^{int}\mathbf{1} = 0$, i.e., no action over the CM acceleration (momentum is conserved in absence of external forces). However, intenal forces **do** affect "shape" evolution (deformation, elastic stress, centripetal stress in rotation), as follows.

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Motion analysis in "mean and variance" (moments) [2]

VARIANCE: Euler tensor (2nd-order mass moment tensor, mass distribution tensor) motion

Consider now $S_{2\times 2} = (X - x_{cm} \cdot \mathbf{1}^T) M (X - x_{cm} \cdot \mathbf{1}^T)^T$. If position were random, this would be the "sample variance-covariance matrix", sort of (weighted by mass here, instead of probability).

This is a "shape" matrix. Indicating the "dispersion" around the center of mass, with directionality information (horizontal vs. vertical dispersion).

- $S = \sum_i m_i (x_i - x_{cm})(x_i - x_{cm})^T$ is a symmetric, positive semidefinite matrix (well, strictly positive definite if mass spans all directions of space, rank 2); all particles on a line $\Rightarrow$ rank 1; all particles converging on a point $S = 0$.
- It has real eigenvalues and orthogonal eigenvectors, which are named as "principal axes".

Notation

For notational brevity, we'll assume $x_{cm} = 0$, i.e., incremental coordinates wrt center of mass, in the sequel.

Momentum virial (outer product): $H = VMX^T = \sum_i m_i v_i x_i^T$.

Force virial: $\mathcal{F} = FX^T = \sum_i F_i x_i^T = \sum_i m_i \frac{d^2 x_i}{dt^2} x_i^T$.

Kinetic outer product tensor: $\mathcal{T} = \frac{1}{2} \sum_i m_i v_i v_i^T$ (symmetric).

Kinetic energy is $trace(\mathcal{T})$.

*Warning: **inertia tensor** in standard textbooks is **not** the mass-distribution "covariance" matrix in the plane, but the "resistance to rotation" around an axis (moment of inertia). I intentionally used the "variance" idea instead of inertia, so your textbook might offer a different formulation. We will derive the "moment of inertia" concept later on, but do NOT confuse the inertia tensor (or inertia matrix) $\boxed{\mathbf{I} = trace(S) \cdot I - S}$ of a **rigid** body with the "mass distribution variance" matrix here.

Equations of motion of the Euler tensor

$$\frac{dS}{dt} = \frac{d}{dt}\left(\sum_i m_i x_i x_i^T\right) = \sum_i m_i v_i x_i^T + \sum_i m_i x_i v_i^T = H + H^T$$

$$\frac{dH}{dt} = \sum_i m_i a_i x_i^T + \sum_i m_i v_i v_i^T = \mathcal{F} + 2\mathcal{T}$$

[1] Indeed, as S is symmetric, $\frac{dS}{dt}$ is also symmetric.

[2] However, H is not necessarily symmetric; neither $\mathcal{F}$ needs to be, as forces are "arbitrary". In fact, the *skew*-symmetric component of H will encode the concept of **angular momentum**, see next.

[3] Recall that $\mathcal{F}$ includes all (interior and exterior) forces. In many physics developments, forces are the gradient of potential fields.

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The equations **cannot** be integrated from state S and H because we need $\mathcal{T}$, and $\dot{\mathcal{T}} = \frac{1}{2}(\sum_i F_i v_i^T + v_i F_i^T)$ needs a "power" tensor. If we knew power tensor (say, a description of the potential, detailed simulation of forces on each particle, etc.), then we could proceed.

Affine velocity field approximation: If we assume that $v_i = Ax_i + v_{cm}$ (affine velocity field), then $H = AS$, and $\dot{S} = AS + SA^T$.
We may write $A = HS^{-1} = (\sum_i m_i v_i x_i^T)(\sum_i m_i x_i x_i^T)^{-1}$ so A is the **least squares fit**, i.e., **best linear estimate of speed given position** (plus center of mass position and speed).
If we assume the least squares fit is **exact**, then $\mathcal{T} = ASA^T = HS^{-1}H^T$.

*It is exact in rigid bodies, expansion/contraction, shear and any linear combination thereof (rotation plus linear deformation, see next slides).
*However, it disregards ellastic higher-order vibration modes, thermal agitation, and fluid turbulence.
Relative acceleration: $a_i - a_{cm} = \dot{A}x_i + A^2 x_i$, then $\sum_i m_i a_i x_i^T = (\dot{A} + A^2)S = \mathcal{F}$, so $\dot{A}$ can be computed if $\mathcal{F}$ were known or, well, a least-squares fit.

The model is not complete

We still cannot fully integrate the matrix motion equations, as $\mathcal{F}$ contains **internal** forces. These are usually unknowns in motion problems but, in fact, they are the dominant contribution governing the evolution of the shape.
Unlike center-of-mass motion, the evolution of shape moments S depends on internal forces. Therefore the equations for S and H are generally not closed: additional information about the body's internal structure (rigidity, elasticity, stress-strain laws, etc.) is required to describe **how internal forces arise from deformation**.

Only in very specific cases (rigid body or affine deformation models $x_i(t) = Q(t)x_{i,0}$, or some astrophysic problems with gravity potentials) internal forces' virial can be explicitly computed from S.

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Symmetric and skew-symmetric components of outer-product momentum virial

Given a matrix A, we can write $A = sym(A) + skew(A)$, with $sym(A) = \frac{1}{2}(A + A^T)$, $skew(A) = \frac{1}{2}(A - A^T)$.

Decomposing $H = sym(H) + skew(H)$ we get:

$$\frac{1}{2}\frac{d^2S}{dt} = \frac{1}{2}\frac{d}{dt}(H + H^T) = \frac{1}{2}(\mathcal{F} + \mathcal{F}^T) + 2\mathcal{T}$$

and the skew-symmetric component of the momentum $K = H - H^T$ (1/2 removed for notational simplicity):

$$\frac{dK}{dt} = \frac{d}{dt}(H - H^T) = \mathcal{F} - \mathcal{F}^T$$

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Colinear interior forces $\Rightarrow$ symmetric virial

If given particle i and particle j, denoting $F_{i\to j}$ the force that particle i makes over particle j, we have that:

$$F_{i\to j} = \kappa_{ij}(x_i - x_j), \qquad F_{j\to i} = -F_{i\to j} = \kappa_{ij}(x_j - x_i)$$

then, the force virial contribution of the interior $i \leftrightarrow j$ interaction to $\mathcal{F}$ is **symmetric**:

$$\begin{aligned} F_{j\to i}\cdot x_i^T + F_{i\to j}\cdot x_j^T &= \kappa_{ij}\cdot\left((x_j - x_i)x_i^T + (x_i - x_j)x_j^T\right) \\ &= \kappa_{ij}\cdot\left(x_j x_i^T + x_i x_j^T - x_i x_i^T - x_j x_j^T\right) \end{aligned}$$

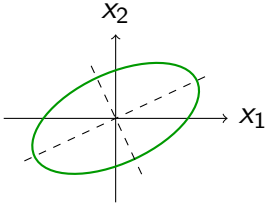
In such a case:

$$\frac{dK}{dt} = \frac{d}{dt}(H - H^T) = \mathcal{F}^{ext} - (\mathcal{F}^{ext})^T$$

Hence, $H - H^T$ (skew-symmetric component of H) is **conserved** if external forces have *null skew-symmetric virial* component. This is the conservation of **angular momentum**; $skew(\mathcal{F}^{ext})$ corresponds to **external torque**.

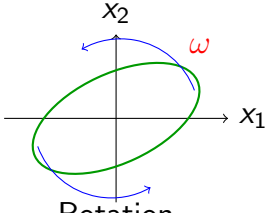
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Interpretation of H: rotation and deformation of an ellipse



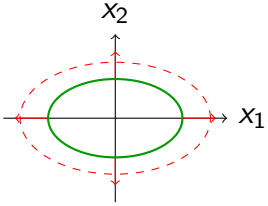
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Principal axes of mass distribution



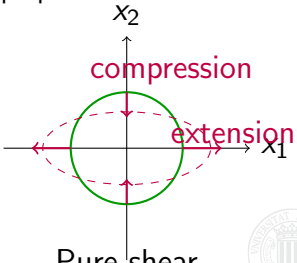
Rotation

Shape preserved



Isotropic expansion

$\lambda_{iso} > 0$



Pure shear

stretch in one direction

compress in the orthogonal one

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Meaning of H in 2D (planar motion): rotation and deformation momentum

Translational motion is summarized in the center of mass equations of motion. In a system with pure translation (constant velocity), $H = \sum_i m_i v x_i^T = v \sum_i m_i x^T = 0$.

So, H encodes "rotation" and "deformation" momentum.
In an "statistical" sense, H is analogous to a **covariance** matrix between velocity and position (mass-weighted and in incremental coordinates wrt center of mass position and mean translational velocity).

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Meaning of H in 2D (planar motion) [1]

In pure rotational motion $v_i = \omega \times x_i = \omega \begin{pmatrix} -x_{i,2} \\ x_{i,1} \end{pmatrix}$. Hence:

$$H = \sum_i m_i \omega \begin{pmatrix} -x_{i,2} \\ x_{i,1} \end{pmatrix} \begin{pmatrix} x_{i,1} & x_{i,2} \end{pmatrix} = \omega \sum_i m_i \begin{pmatrix} -x_{i,2} \cdot x_{i,1} & -x_{i,2}^2 \\ x_{i,1}^2 & x_{i,1} \cdot x_{i,2} \end{pmatrix}$$

$$= \omega \cdot \begin{pmatrix} -s_{12} & -s_{22} \\ s_{11} & s_{12} \end{pmatrix} = \omega \cdot \begin{pmatrix} -s_{12} & \frac{1}{2}(s_{11} - s_{22}) \\ \frac{1}{2}(s_{11} - s_{22}) & s_{12} \end{pmatrix} + \underbrace{\omega \cdot \begin{pmatrix} 0 & \frac{-1}{2}(s_{11} + s_{22}) \\ \frac{1}{2}(s_{11} + s_{22}) & 0 \end{pmatrix}}_{\text{conserved if } skew(\mathcal{F}^{ext}) = 0}$$

Note that $I_z = trace(S) = s_{11} + s_{22}$ is invariant under rotations, as it its the sum of eigenvalues and $RXMX^TR^T = RSR^{-1}$ has the same eigenvalues as S, it's only the eigenvectors which rotate.

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Meaning of H in 2D (planar motion) [2]

Asume "principal axis" as reference system, diagonal S, $s_{12} = s_{21} = 0$, for simplicity. The momentum virial $H = sym(H) + skew(H)$, is decomposed in two components, a **rotation transport** one and a **deformation** one.

Rotation transport:

- Rotational transport momentum symmetric component (shape transport): $H_{rot} + H_{rot}^T = \omega \cdot \begin{pmatrix} 0 & s_{11} - s_{22} \\ s_{11} - s_{22} & 0 \end{pmatrix}$
- A skew symmetric angular momentum: $\begin{pmatrix} 0 & -L_z \\ L_z & 0 \end{pmatrix}$, with $L_z = \sum_i m_i(x_{i,1}v_{i,2} - x_{i,2}v_{i,1})$ being the **scalar** angular momentum. In case motion is a pure rotation, it is: $\omega \begin{pmatrix} 0 & -s_{11} - s_{22} \\ s_{11} + s_{22} & 0 \end{pmatrix} = \omega \begin{pmatrix} 0 & -I_z \\ I_z & 0 \end{pmatrix}$

*Well, we *can* use it to define a "mean" angular speed ω of a "non-rigid" ensemble.

The concept of external torque

We had $$\frac{dK}{dt} = \frac{d}{dt}(H - H^T) = \mathcal{F}^{ext} - (\mathcal{F}^{ext})^T$$

- The left hand side is: $\frac{d}{dt}\begin{pmatrix} 0 & -L_z \\ L_z & 0 \end{pmatrix}$, with $L_z = \sum_i m_i \left(x_{i,1}v_{i,2} - x_{i,2}v_{i,1}\right)$ known as **angular momentum**.

- L_z may be used to define a "mean angular velocity" $\omega = L_z/I_z$, $I_z = s_{11} + s_{22}$, $L_z = \omega I_z$, which will be the "true" angular velocity if motion were a **pure rigid rotation**.

- The right hand side is: $\begin{pmatrix} 0 & -T_z \\ T_z & 0 \end{pmatrix}$, with $T_z = \sum_i x_{i,1}F^{ext}_{i,2} - x_{i,2}F^{ext}_{i,1}$ known as **external torque**. This is the "outside thing" that can change angular momentum.

Scalar equations of 2D rotation around center of mass

Rigid and Non-Rigid motion: $\frac{dL_z}{dt} = \frac{d}{dt}(\omega I_z) = T_z$

*In general, the change of I_z with time will involve the symmetric component of momentum H and its second derivative will involve internal forces.

Rigid body only ($I_z = \sum_i m_i(x_{i,1}^2 + x_{i,2}^2)$ is constant):

$$I_z \frac{d\omega}{dt} = T_z \qquad \text{, i.e., } \qquad \frac{d\omega}{dt} = \frac{1}{I_z} \cdot T_z$$

Hint 1: Why developing things in "matrix world" with skew-symmetric matrices? Because the ideas can be generalized to the "Lie group" of planar rotation and its "Lie algebra" of changes in rotation.

Hint 2: the 3D case will be also based on a skew-symmetric 3×3 matrix and 3 scalar equations arising from its off-diagonal terms.

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Meaning of H in 2D (planar motion) [3]

The remaining symmetric components of H verify that

$$\frac{d}{dt}(\frac{1}{2}(H + H^T)) = \mathcal{F}^{sym} + 2\mathcal{T} = \mathcal{F}^{sym} + \sum_i m_i v_i v_i^T$$

can be interpreted of as:

- Momentum tensor change because of "free particle motion": $2\mathcal{T}$ [no forces: same velocity, different position after some time]
- Cumulative "stress" arising from **external** and **internal** forces in $\mathcal{F}^{sym}$.

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*The "principal axes" of S, or $sym(H)$, or $\mathcal{F}^{sym}$ won't in general coincide.

In the principal axis of $\mathcal{F}^{sym} = U\Lambda U^T$, its eigenvalues (real) p_1 and p_2 in the diagonal of Λ may be understood (in an "average" sense, as we are summing over all particles) as:

- $\lambda_{iso} = \frac{p_1+p_2}{2}$ the **isotropic** stress producing compression/expansion "pressure", depending on the sign of λ_{iso} (changes volume).
- $\lambda_{dev} = \frac{p_1-p_2}{2}$ the **deviatoric** stress component (shape-changing).

$$\mathcal{F}^{sym} = U \cdot \left(\lambda_{iso} \cdot \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \lambda_{dev} \cdot \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \right) \cdot U^T$$

The F^{sym} eigenvectors indicate the directions of maximum (in average) expansion/contraction effect of forces rather than the directions of maximum/minimum mass distribution (eigenvectors of S).

*Details on continuum mechanics, stress analysis, etc. are out of the scope of this material.

Conclusions/summary

- Instead of analyzing motion of each "individual particle" in a system, we can analyze "mean" (center of mass, mean linear momentum) and "variance" (Euler tensor, moments of inertia).
- Analysis involves position×position (variance mass distribution), velocity×position (momentum tensor), and velocity×velocity (kinetic tensor), plus their time derivatives.
- Momentum decomposes in translational (vector), plus matrix components: angular momentum (changed by external torque) + symmetric momentum (changed by pressure and deviatoric stresses).
- **Two conservation laws:** linear momentum vector if $\sum F^{ext} = 0_{2\times 1}$, skew-symmetric angular momentum tensor (i.e., scalar angular momentum, actually) if $\sum_i (F_i^{ext} x_i^T - x_i (F^{ext})^T) = 0_{2\times 2}$ (i.e., $\sum_i F_{i,1} x_{i,2} - x_{i,1} F_{i,2} = 0$).

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Usage en engineering

- In aerospace/robotics/control engineering usually **rigid** body motions are only considered in a first step of the analysis.
- Expansion/contraction/shear are more a matter of structural calculus, continuum mechanics, to decide if a robot/aircraft/spacecraft fails.
- In rigid body motion we just assume that "deforming" momentum/forces will be zero; the rigidity constraints enforce these deformation modes to remain zero; the evolution of S and H is entirely explained by rotation, internal forces should keep the structure in shape.

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What's missing yet

- Specialization to rigid body dynamics
- Extension to 3D motion
- Continuum mechanics, stress tensor, elasticity, rate-of-deformation tensor $D = \frac{1}{2}\left(\nabla v + \nabla v^T\right)$, internal vibration modes, etc.
- Affine body deformations $x(t) = \mathcal{A}(t)x(0)$ and an elastic potential energy $U(\mathcal{A})$, $\dot{x} = \dot{\mathcal{A}} \cdot x(0)$, as an intermediate between "rigid body" and, say, "full finite-element analysis"; a sort of "linear least squares" approximation of elasticity.
- In Physics, some refinements on the issues discussed here end up being the *scalar virial theorem* or the *tensor virial theorem*.
- The part of the kinetic energy that is not "correlated" with translational or rotational velocity are internal vibration modes (or temperature kinetic energy if we enter the realm of thermodynamics and statistical physics, $N \to \infty$).