Modelling of a heating element as a 1st order system assuming exponential profile of internal temperature

Video-presentation: http://personales.upv.es/asala/YT/V/term1expEN.html

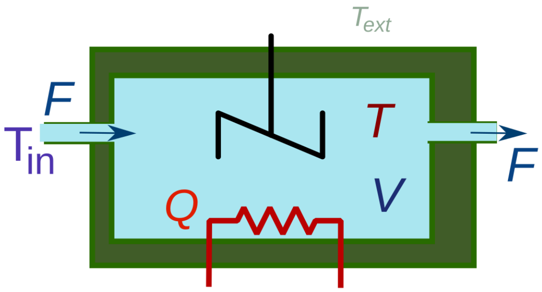
Comments, PDF notes and erratum in the link in the video's description

Objectives: model a tank as a 1st-order dynamical system, but considering it "tubular", taking into account the steady-state exponential temperature profile arising from well-known stationary PDE solutions.

Table of Contents

First-principle model

If stirred, all temperature inside the volume is the same... if NOT stirred (the case here), and "long/ tubular", we'll look at it later on.



- Inputs:

```
syms F real %input flow
syms Tin real %input fluid's temperature
syms Q real %total heating power
```

- Parámeters (constant):

```
syms V real% total volume
syms rho real %density
syms kappa real % conduction heat transfer coefficient
```

```
syms Ce real %specific heat
```

NOTE: Specific heat constant will be renamed to C_e for consistency with my notes in Spanish language, sorry for the inconvenience; in other English-language material it was "c".

- State variable:

```
syms T  real %mean temperature of the fluid inside the (tubular) tank
syms dTdt real % time derivative
```

Mass balance is trivial: flow entering the tank is the same as the flow exiting the tank.

Power (rate of change of energy) balance is

" *rate of change of energy inside control volume* $\left(\frac{dE}{dt} = MC_e\frac{dT}{dt}\right)$, mass is constant inside the control volume so dM/dt=0, mass change needs not considered inside the said volume.

= *net heat power exchange with the outside environment* $(Q - \kappa T)$

+ *total energy entering the control volume per unit time due to incoming fluid* $+\frac{dM_{in}}{dt}C_eT_{in} = (+F\rho C_eT_{in})$

- *total energy per unit time leaving the control volume due to outgoing fluid* $-\frac{dM_{out}}{dt}C_eT_{out} = (-F\rho C_eT_{out})$"

which is written as:

$$\underbrace{V\rho}\, C_e\frac{dT}{dt} = \underline{F\rho}C_eT_{in} - \underline{F\rho}C_eT_{out} - \kappa T + Q$$

Nota: $V\rho = Mass$, $F\rho =$mass flow $\dot{m}$; the term $\dot{m}cT$ has dimensions of power (enthalpy flow rate), we have incoming power $\dot{m}cT_{in}$ and outgoing power $\dot{m}cT_{out}$.

1.) Perfectly-stirred tank

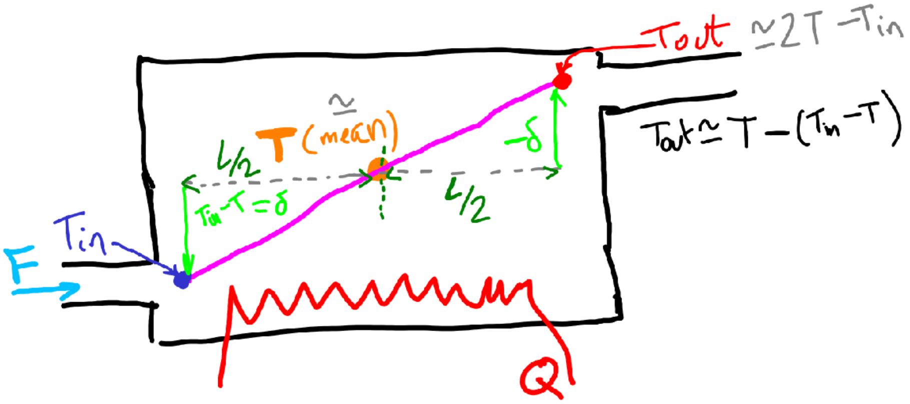
We may assume $T_{out} \equiv T$, getting a model:

$$\underbrace{V\rho}\, C_e\frac{dT}{dt} = \underline{F\rho}C_eT_{in} - \underline{F\rho}C_eT - \kappa T + Q$$

Normalised representation:

- state equation $\frac{dT}{dt} = -\frac{F}{V} \cdot (T - T_{in}) - \frac{\kappa}{V\rho C_e} T + \frac{1}{V\rho C_e} Q$,
- output $T_{out} = T$.

2.) Imperfect stirring, assuming linear temperature profile



if the "mean" temperature is $T = 0.5T_{in} + 0.5T_{out}$, solving for T_{out} we get: $T_{out} = 2T - T_{in}$

The model in such a case has twice as fast the transport dynamics:

$$\underbrace{V\rho}_{} C_e \frac{dT}{dt} = F\rho C_e T_{in} - F\rho C_e T_{out} - \kappa T + Q = F\rho C_e T_{in} - F\rho C_e(2T - T_{in}) - \kappa T + Q = 2F\rho C_e(T_{in} - T) - \kappa T + Q$$

Normalised representation:

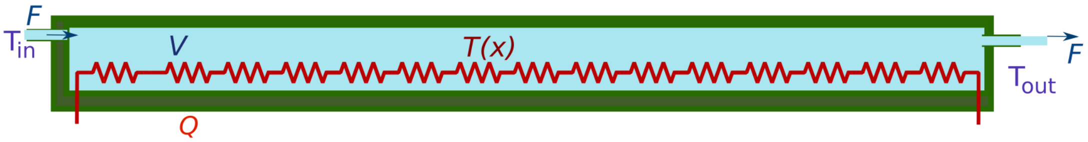
- state equation:

$$\frac{dT}{dt} = -2\frac{F}{V} \cdot (T - T_{in}) - \frac{\kappa}{V\rho C_e} T + \frac{1}{V\rho C_e} Q$$

- output equation:

$$T_{out} = 2T - T_{in}$$

3) More general Imperfect stirring (or absence thereof): exponential profile



The equilibrium solution to the one-dimensional PDE associated to a tubular heat exchanger with constant environment's temperature is $T_{eq}(x) = T_{in} \cdot e^{-\lambda \cdot x} + (1 - e^{-\lambda \cdot x})\overline{\kappa}^{-1} \cdot \overline{Q}_{eq}$.

If we assume $T(x) = Ae^{-\lambda x} + B$, we have $T_{in} = T(0) = A + B$, $T_{out} = T(L) = Ae^{-\lambda L} + B$, con lo que

```
syms ti to a b el real
M=[ti==a+b; to==a*el+b]
```

```
M =
```

$$\begin{pmatrix} \mathrm{ti} = a + b \\ \mathrm{to} = b + a\,\mathrm{el} \end{pmatrix}$$

```
solve(M,{a,b})
```

```
ans = struct with fields:
    a: -(ti - to)/(el - 1)
    b: -(to - el*ti)/(el - 1)
```

These are the expressions, thus, for A and B:

$$A = \frac{T_{in} - T_{out}}{1 - e^{-\lambda L}}, \qquad B = \frac{T_{out} - e^{-\lambda L} T_{in}}{1 - e^{-\lambda L}}$$

Let us define the mean temperature $\overline{T} = \frac{1}{L}\int_0^L T(x)\,dx$; stored energy (no phase changes, zero at outside temperature) is $E = V\rho C_e \overline{T}$.

Indeed $E = \int_0^L \rho C_e T(x) S dx$, hence, $E = (S \cdot L \cdot C_e \cdot \rho) \cdot \frac{1}{L}\int_0^L T(x)dx = V\rho C_e \overline{T}$.

Carrying out the integral, we have:

$$\overline{T} = \frac{A}{\lambda L}(1 - e^{-\lambda L}) + B = \frac{T_{in} - T_{out}}{\lambda L} + \frac{T_{out} - e^{-\lambda L} T_{in}}{1 - e^{-\lambda L}}$$

Remark: as a particular case, with no heating Q and assuming heat exchanger is in steady-state, then $T_{out} = e^{-\lambda L} T_{in}$ (recall that outside temperature is set to zero in our reference system), so $B = 0$ and $\overline{T} = \frac{T_{in} - T_{out}}{\lambda L}$ is the so-called LMTD (Logarithmic mean temperature difference), $\overline{T} = \frac{T_{in} - T_{out}}{\ln T_{in} - \ln T_{out}}$.

The LMTD plays a KEY role in many (stationary) heat-exchanger calculations.

Common denominator gives:

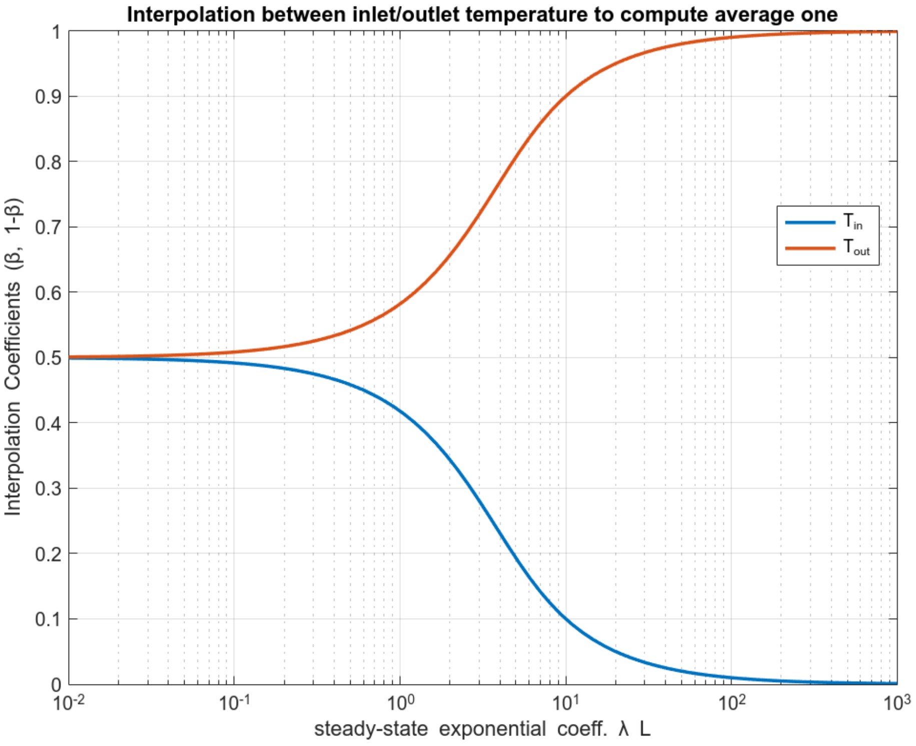
$$\overline{T} = \frac{T_{in}(1-(1+\lambda L)e^{-\lambda L}) + T_{out}(\lambda L - 1 + e^{-\lambda L})}{\lambda(1-e^{-\lambda L})\cdot L}$$

And, with some operations, we can realise that coefficients for each temperature add one:

$$\overline{T} = (1-\beta)T_{in} + \beta T_{out}$$

so the mean temperature is an ***interpolation*** of the inlet and outlet ones, being the interpolation weight $\beta(\lambda L) = \dfrac{\lambda L - (1-e^{-\lambda L})}{\lambda L(1-e^{-\lambda L})} = \dfrac{1}{\dfrac{1}{1-e^{-\lambda L}} - \dfrac{1}{\lambda L}}$. Different values for λL yield:

```
lamLr=logspace(-2,3,80);
explamL=exp(-lamLr);
beta=(lamLr-1+explamL)./(lamLr.*(1-explamL));
semilogx(lamLr,[1-beta; beta],LineWidth=2), grid on, legend("T_{in}","T_{out}",Location
ylabel('Interpolation Coefficients (\beta, 1-\beta)')
title("Interpolation between inlet/outlet temperature to compute average one"), xlabel
```



Limits

- when $\lambda \to \infty$ we get: $\bar{T} \to T_{out}$, $\beta = 1$

- when $\lambda \to 0$, we set $e^{-\lambda L} \approx 1 - \lambda L + \lambda^2 L^2/2$

$$\bar{T} = \frac{T_{in}(1-(1+\lambda L)(1-\lambda L+\lambda^2L^2/2)) + T_{out}(\lambda L - 1 + (1-\lambda L+\lambda^2L^2/2))}{\lambda(1-(1-\lambda L+\lambda^2L^2/2))\cdot L}$$

So taking limits, we can get:

$$\bar{T} = \frac{T_{in}\lambda^2L^2/2 + T_{out}\lambda^2L^2/2}{\lambda^2L^2} = (T_{in}+T_{out})/2 \text{, i.e., } \beta = 0.5.$$

Power balance in 1st-order model:

In the case $T = (1-\beta)T_{in} + \beta T_{out}$, with $0.5 \le \beta \le 1$, we removed the "bar" notation in mean temperature, for notational simplicity.

Solving for temperature at exit side: $T_{out} = \frac{1}{\beta}T - \frac{1-\beta}{\beta}T_{in}$

$$\rho C_e \frac{dT}{dt} = F\rho C_e T_{in} - F\rho C_e T_{out} - \kappa T + Q = F\rho C_e T_{in} - F\rho C_e(\frac{1}{\beta}T - \frac{1-\beta}{\beta}T_{in}) - \kappa T + Q = \frac{1}{\beta}F\rho C_e(T_{in} - T) - \kappa T +$$

- This will be the state equation, with a slightly faster transport-related dynamics than that of the perfect mixing:

$$\frac{dT}{dt} = -\frac{1}{\beta}\frac{F}{V}\cdot(T - T_{in}) - \frac{\kappa}{V\rho C_e}T + \frac{1}{V\rho C_e}Q$$

- This is the new output equation:

$$T_{out} = \frac{1}{\beta}T - \frac{1-\beta}{\beta}T_{in}$$

*Note that if β is set up to "preserve stationary regime" it will be a function of physical parameters (heat conductivity, dimensions) as well as the input flow and resistor heat.

Of course, in a generic simulation with varying Q, F, T_{in} such optimal β would change with time... unless we assume som operating point so all are close to some nominal values, so we may have an idea for β and furthermore, we might wish to linearise the model.

Linearisation/ Transfer Function

Linearising the product of flow times temperatures around an operating point, we get:

$$\frac{dT}{dt} \approx -\frac{1}{\beta}\frac{F_0}{V} \cdot (T - T_{in}) - \frac{1}{\beta}\frac{F}{V} \cdot (T_0 - T_{in,0}) - \frac{\kappa}{V\rho C_e}T + \frac{1}{V\rho C_e}Q$$

being $\mu = -(T_0 - T_{in,0})/V$, negative if mean temperature is higher than the inlet one at the operating point (heating on), positive when mean temperature is lower than the input temperature (heat exchanger with no internal heating resistor).

Carrying out some algebraic operations on the Laplace transform under zero initial conditions, we would get:

$$T(s) = \frac{\frac{1}{V\rho C_e}Q + \frac{F_0}{\beta V}T_{in} + \frac{\mu}{\beta}F}{s + \frac{\kappa}{V\rho C_e} + \frac{F_0}{\beta V}}$$

$$T_{out}(s) = \frac{\frac{1}{\beta V\rho C_e}Q + \frac{F_0}{\beta^2 V}T_{in} + \frac{\mu}{\beta^2}F}{s + \frac{\kappa}{V\rho C_e} + \frac{F_0}{\beta V}} - \frac{s + \frac{\kappa}{V\rho C_e} + \frac{F_0}{\beta V}}{s + \frac{\kappa}{V\rho C_e} + \frac{F_0}{\beta V}}(\frac{1}{\beta} - 1)T_{in}$$

Final result is:

$$T_{out}(s) = \frac{\frac{1}{\beta V\rho C_e}}{s + \frac{\kappa}{V\rho C_e} + \frac{F_0}{\beta V}} \cdot Q(s) + \frac{\frac{\mu}{\beta^2}}{s + \frac{\kappa}{V\rho C_e} + \frac{F_0}{\beta V}} \cdot F(s) + \frac{(s + \frac{\kappa}{V\rho C_e})(1 - \frac{1}{\beta}) + \frac{F_0}{\beta V}}{s + \frac{\kappa}{V\rho C_e} + \frac{F_0}{\beta V}} \cdot T_{in}(s)$$

Obviously, transient will not be too accurate (it's infinite-order PDE), but, well, the choice of β may help fitting the steady-state conditions to that of the PDE. Non-minimum phase components in T_{in} are reminiscent of Padé approximations of the time delay.

We'll see this in simulations in other materials.