

Ellipsoids, relation with positive-definite matrices: basic definitions

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Video presentations:

<http://personales.upv.es/asala/YT/V/ellip1EN.html>

<http://personales.upv.es/asala/YT/V/ellip2EN.html>

<http://personales.upv.es/asala/YT/V/ellip3EN.html>



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Outline

Motivation:

Ellipsoids are geometric bodies associated with positive definite symmetric matrices that appear in a variety of problems, control theory being one of them.

Objectives:

Understand their definition, importance in applications and basic properties.

Contents:

Definitions. Applications. Alternative representations. Diagonalization, SVD, other properties.



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Basic definitions (1): spheres

An n -dimensional sphere of unit radius, centered at the origin, is defined as $\{x \in \mathbb{R}^n : \sum_{i=1}^n x_i^2 \leq 1\}$. Matrix notation:

$$x^T x = x^T \cdot I \cdot x \leq 1.$$

An n -dimensional sphere of radius ρ centered at the origin changes to $\sum_{i=1}^n x_i^2 \leq \rho^2$, that is, $\sum_{i=1}^n \left(\frac{x_i}{\rho}\right)^2 \leq 1$.

Matrix notation: $(x/\rho)^T (x/\rho) = x^T \cdot \rho^{-2} I \cdot x \leq 1$.



Basic definitions (2)

Ellipsoid (aligned with axis): If all points in unit sphere are mapped to $\tilde{x}_1 = 2x_1$, $\tilde{x}_2 = x_2$, \dots (we “double” the size in the x_1 dimension), then $\frac{\tilde{x}_1^2}{2^2} + \tilde{x}_2^2 + \dots + \tilde{x}_n^2 \leq 1$.

If all points in unit sphere are mapped to $\tilde{x}_1 = \sigma_1 x_1$, $\tilde{x}_2 = \sigma_2 x_2$, \dots , then: $\sum_{i=1}^n \frac{\tilde{x}_i^2}{\sigma_i^2} \leq 1$.

In matrix notation, $\tilde{x} = \Sigma x$, Σ is diagonal;

$\tilde{x}^T \Sigma^{-2} \tilde{x} = \tilde{x}^T \Lambda \tilde{x} \leq 1$, with $\Lambda = (\Sigma^2)^{-1}$ a diagonal matrix (with positive elements $\lambda_i = \sigma_i^{-2}$).



Motivation (control applications)

- Bounding the “norm” of signals $\|\xi\|^2 = \sum_i \xi_i^2 = \xi^T \xi \leq \gamma$.
 - Errors must be small, we wish $e^T e \leq \gamma$ with γ being “the smallest possible” ... Control actions must not saturate...
 - Once we have multiple controlled/manipulated variables in different units, we have expressions such as $u_1^2 + (u_3/5)^2 = u^T P u \leq \gamma$, $P = \text{diag}(1, 1/5^2)$, as ‘ γ level sets’ in weighted least squares.
 - In order for $y = Cx$ to verify $y^T y \leq 1$, state must verify $x^T (C^T C) x \leq 1$, with non-diagonal $P = C^T C$...
- Robotics: *manipulability, force, compliance ellipsoids.*
- [Advanced:] robust control, linear matrix inequalities (LMI).



Motivation (other than control)

- Pure geometrical interest (foci, volume, semiaxis, ...)
- Astronomy (Kepler laws), ...
- Relationship with least squares (spheres) and weighted least squares (ellipsoids)
- Confidence ellipsoid in multivariate statistics, generalising “confidence interval” in single-variable statistics
- “Stress/strain ellipsoid” in elasticity in material science.



Definition: **generic ellipsoid** (centered)

If we rotate an axis-aligned ellipsoid (rotation matrix $R \equiv$ orthogonal matrix $R^T R = I$, $R^{-1} = R^T$, $\det(R) = 1$), then $\tilde{x} = R \Sigma x$, $x = \Sigma^{-1} R^T \tilde{x}$, so $x^T x \leq 1$ is written in new coordinates as: $\tilde{x}^T R \Sigma^{-2} R^T \tilde{x} \leq 1$.

Diagonalization of an arbitrary positive-definite matrix P allows writing $P = R \Lambda R^T$ (R orthogonal, Λ diagonal), motivating the general definition:

An ellipsoid (centered at the origin) is the geometric body determined by a quadratic positive-definite form as:

$$\mathcal{E}_P = \{x \in \mathbb{R}^n : x^T P x \leq 1\}$$

Ellipsoid centered at $x_0 \neq 0$

If an ellipsoid is not centered at the origin, its definition is

$$\mathcal{E}_{(x_0, P)} = \{x \in \mathbb{R}^n : (x - x_0)^T P (x - x_0) \leq 1\}$$

In control, the origin $x_0 = 0$ is the operating point around which we wish state to lie; we don't usually consider $x_0 \neq 0$, it is just a straightforward change of variable to “increments”. Levels different to 1 are also usually disregarded as, for instance, $x^T P x \leq 2$ coincides with $x^T (P/2) x \leq 1$.

In statistics it is also customary to trivially translate the data to convert the “sample average” to zero and speak about increments around such mean; this is called “centering” in statistical jargon... Well, sample average may not coincide with “true” mean, so it might be not so “trivial”.

Axis, diagonalization

Given $x^T P x \leq 1$, diagonalizing $P = V D V^T$, then V is orthogonal (rotation/reflection) and D is diagonal.

The rotation change of variable $\tilde{x} = V^T x$ results in $x = V \tilde{x}$, $x^T P x = \tilde{x}^T V^T P V \tilde{x}$, but $V^T (V D V^T) V = D$ so $\tilde{x}^T D \tilde{x} \leq 1$, is an axis-aligned ellipsoid.

The semi-axis lengths of such an ellipsoid are $\sigma_i = 1/\sqrt{d_{ii}}$.



Other representations of ellipsoids

- Inverted-matrix form: $\mathcal{E}_{Q^{-1}} = \{x : x^T Q^{-1} x \leq 1\}$,
 $Q = P^{-1}, \equiv \{x^T P x \leq 1\}$

► P is “**inversely** proportional to square of size”,

Eigenvectors are directions of semiaxis; their length is $1/\sqrt{\text{eigenvalues}(P)}$

► Q is “**directly** proportional to square of size” ...

Eigenvectors are directions of semiaxis; their length is $\sqrt{\text{eigenvalues}(Q)}$

Example:

Ellipsoid $x^T \cdot 9 \cdot x \leq 1$ has radius $1/3$.

Ellipsoid $x^T \cdot (25)^{-1} \cdot x \leq 1$ has radius 5.



Degenerate ellipsoids, cylinders and hyperboloids

- $x^T P x \leq 1$: if P singular (one zero eigenvalue, rest ≥ 0), equation of a cylinder with elliptical base ($x^2 + z^2 \leq 1$ is a cylinder with axis in y direction)
- $x^T Q^{-1} x \leq 1$ if Q singular, it describes a lower-dimensional ellipsoid $\{x^2 + z^2 \leq 1, y = 0\}$.
 *We cannot “write” Q^{-1} , because it does not exist: consider it as an “informal” expression (formally, it should entail limits).
- If P has negative eigenvalues, hyperboloids may arise $x^2 - y^2 \leq 1$.

Degenerate cases are not usually seen in control and statistics... first and second might indicate “conceptual error” (lack of controllability, observability; statistical determinism)... Hyperbolic geometry is of interest in Physics (special relativity).