Dynamics of planar 2DoF movement (point mass) in intrinsic coordinates

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Video-presentation at: http://personales.upv.es/asala/YT/V/intrid1EN.html

Outline

Motivation:

The velocity (gyroscope, speedometer) and acceleration (accelerometer) sensors of a car or plane, or its driver/pilot, sense things in "body frame". They move forward or turn. When facing a curve, it doesn't matter if you enter it facing East or South, it only matters how it "curves". This will apply in vehicle dynamics, aerial robotics, etc.

Objectives:

Establish the equations of motion of a point (2GL) when velocity and acceleration are measured in a coordinate base that moves with that point ("intrinsic").

Contents:

Tangent and normal vectors. Tangential and normal acceleration. Dynamics (forces) in intrinsic coordinates. Curvature. conclusions.

Kinematics in Frenet coordinates (tangent/normal, coord. intrinsic)

Tangent vector: $\vec{v}(t) := \nu(t)\vec{T}(t)$, $\vec{T}(t) = (\cos\theta(t), \sin\theta(t))$.

We define $\vec{T}$ as the direction of the velocity vector (unitary). The angle θ must be measured with respect to an "extrinsic" fixed axis (linear and angular positions are measured with respect to an external "inertial" reference system).

Abusing notation, we also call $\nu(t)$ velocity (tangential), $\nu = ||\vec{v}||$; assuming $\nu > 0$. It is the arc length traveled per unit of time, $\frac{ds}{dt}$.

In "mechanics", going along the curve faster will require more acceleration; in pure "geometry", it is usual to assume $\nu = 1$ (defining "normalized" curvature magnitudes, independent of the speed at which the curve is traversed).

Normal vector at a point to a trajectory

Normal "to the left" (counterclockwise, port side):

$\frac{d\vec{T}}{dt} = (-\sin\theta, \cos\theta)\cdot\frac{d\theta}{dt} := \frac{d\theta}{dt}\vec{N}_L$, perp. to $\vec{T}$.

Frenet Normal Vector:

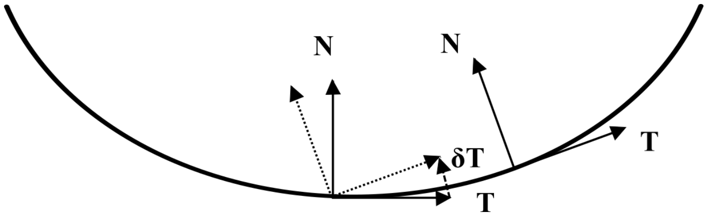
If $\frac{d\theta}{dt}$ is negative (rotates to the right, clockwise), what is called the (Frenet) "normal vector" points towards the side it is being rotated to.

$$\frac{d\vec{T}}{dt} := \left|\frac{d\theta}{dt}\right|\cdot\vec{N}$$

where $\vec{N}$ is $\vec{N}_L := (-\sin\theta, \cos\theta)$ if turning "counterclocwise" and $-\vec{N}_L = (\sin\theta, -\cos\theta)$ it turning "clockwise".

Physics text use $\vec{N}$ by default, but absolute value of derivatives is clumsy to handle in differential equations, so we will use $\vec{N}_L$ later on.

"Intrinsic" coordinate basis: The vectors $\vec{T}$ and $\vec{N}$ form an orthonormal basis... note, they "move" so it is not "inertial" for the equations of mechanics.



*In addition to moving, vector $\vec{N}$ changes sign at the inflection points, where $\frac{d\theta}{dt}$ changes sign.

Acceleration in Frenet intrinsic coordinates (tangent/normal)

Acceleration $\vec{a} := \frac{d\vec{v}}{dt}$ is another vector, so we define its tangential and normal components with: $\vec{a} = \frac{d\vec{v}}{dt} := a_T\vec{T} + a_{N_L}\vec{N}_L = a_T\vec{T} + a_N\vec{N}$

Derivative of the product $\vec{v} = \nu\vec{T}$:

$$\frac{d\vec{v}}{dt} = \frac{d\nu}{dt}\vec{T} + \nu \cdot \frac{d\vec{T}}{dt} = \frac{d\nu}{dt}\vec{T} + \nu\frac{d\theta}{dt}\vec{N}_L = \frac{d\nu}{dt}\vec{T} + \nu\left|\frac{d\theta}{dt}\right|\vec{N}$$

Hence:

$$a_T = \frac{d\nu}{dt} \qquad a_{N_L} = \nu\frac{d\theta}{dt}, \qquad a_N = \nu\left|\frac{d\theta}{dt}\right|$$

Dynamics, Forces

Under the action of forces (assuming constant mass), we have $m\vec{a} = m\frac{d\vec{v}}{dt} = \vec{F}$, understanding $\vec{v}$ and $\vec{F}$ as vectors existing "by themselves", not yet associated with any "base" with "coordinates".

Projecting over $\vec{T}$ and $\vec{N}$ (or $\vec{T}$ and $\vec{N}_L$), results in $m \cdot a_T = F_T$, $m \cdot a_N = F_N$ (or, preferably, $m \cdot a_{N_L} = F_{N_L}$, to avoid absolute values in the ODEs below). So, from previous slide:

$$m\frac{d\nu}{dt} = F_T \qquad m\nu\frac{d\theta}{dt} = F_{N_L}$$

Reminder: non-inertial reference frame

If $\vec{T}$ and $\vec{N}_L$ were constant ("inertial" orthonormal frame), then Newton's law would be $m \cdot \frac{d}{dt}(v_T) = F_T$, $m \cdot \frac{d}{dt}(v_{N_L}) = F_{N_L}$, being v_T and v_{N_L} the components over $\vec{N}_L$ and $\vec{T}$ of the velocity vector. Obviously, the second equation is **WRONG** because v_{N_L} is always zero in "intrinsic" coordinates (by definition), and so it will be its derivative.

That is why Newton's laws $\vec{F} = M\vec{a}$ are not correctly expressed in this mobile coordinate system without the manipulations in previous slides.

Equations with position

If now we use an "extrinsic" inertial coordinate system (position $\vec{p}(t) = (x(t), y(t))$ is, say, given by GPS) for the position (x, y, θ), then we will have, in normalized state-space representation:

$$\frac{dx}{dt} = \nu \cos\theta$$

$$\frac{dy}{dt} = \nu \sin\theta$$

$$\frac{d\theta}{dt} = \frac{F_{N_L}}{m \cdot \nu}$$ "Normal" acceleration dynamics (signed)

$$\frac{d\nu}{dt} = \frac{F_T}{m}$$ Tangential acceleration dynamics

Navigation+AHRS sensors:
(x, y) will be read by GPS, θ (body frame) by compass/magnetometer/gyro, $\frac{d\theta}{dt}$ by gyroscope, $\frac{d\nu}{dt}$ by accelerometer (+weight vector).

Conclusions

- The dynamics of a point (2 degrees of freedom), in "inertial" coordinates, is of order 4, states (x, v_x, y, v_y), i.e. $(\dot{x} = v_x, \dot{v}_x = F_x/m, \dot{y} = v_y, \dot{v}_y = F_y/m)$.
- Said dynamics is also of order 4 in "intrinsic" coordinates, as we have obtained here, using linear velocity ν and angular velocity $\frac{d\theta}{dt}$. These coordinates can be useful in vehicle and aircraft dynamics.

["path" reference frame]

***Note:** Angle θ is the angle of the tangent to the path of the "CoG point"; a rigid body in planar mov. has 1 extra GL (2GL of mov. cdg + 1GL angular).

General case in vehicle dynamics considers at least 3 reference frames: "inertial", "body" and "path" frame.

- An aircraft, specially during maneuvers, does not have to be "aligned" with its velocity path vector `https://youtu.be/PXU5_K1xioI` ; neither is a rally car skidding around a curve `https://youtu.be/4pKKqguWTb0` .