Modelling of a tubular heater with partial differential equations: PDE solution for constant flow

Presentation in video: http://personales.upv.es/asala/YT/V/termedpsolEN.html

*Code/PDF/notes/erratum in link in the video's description.

Goal: Understanding how to solve (obtaining the transfer function for each of the inputs) the heat exchanger PDE with the Laplace transform method on the "time-dependent" side.

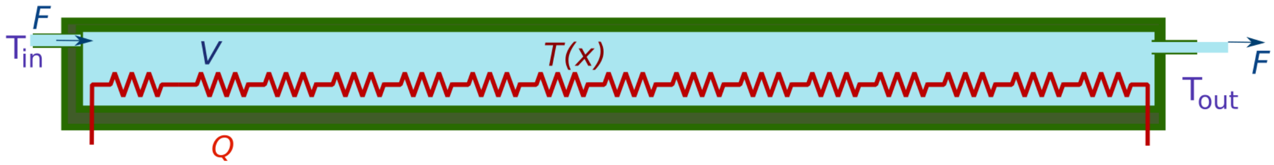

Table of Contents

Preliminaries: review of the PDE model

This is the PDE that describes the tubular heat exchanger under 1D laminar flow assumptions:

$$\frac{\partial T}{\partial t} = -\frac{1}{S}F\frac{\partial T}{\partial x} - \frac{\bar{\kappa}}{S\rho C_e}T + \frac{\bar{Q}}{S\rho C_e}$$

Details are described in other materials.

F is the volumetric flow, $\bar{Q}$ is heat power per unit length, S is the cross-section area, ρ the fluid density, C_e the specific heat K/(KgJ), $\bar{\kappa}$ is the conduction heat transfer per unit length.

There are a couple of interesting particular cases

- With $\bar{Q} = 0$ y $\bar{\kappa} = 0$ we have a "transport delay" PDE, $\frac{\partial T}{\partial t} = -v\frac{\partial T}{\partial x}$, being $v = \frac{F}{S}$ the linear speed of said transport.

- In a stationary case, setting $\frac{\partial T}{\partial t} = 0$, if we denote the steady-state solution as $T_{eq}(x)$, this results in $\frac{\partial T_{eq}}{\partial x} = -\frac{\bar{\kappa}}{F\rho C_e}T_{eq} + \frac{1}{F\rho C_e}\bar{Q}_{eq}(x)$, so, with constant heating power along all the piping, defining $\lambda = \frac{\bar{\kappa}}{F\rho C_e}$ we would have:

$$T_{eq}(x) = T_{in} \cdot e^{-\lambda \cdot x} + (1 - e^{-\lambda \cdot x})\overline{\kappa}^{-1} \cdot \overline{Q}_{eq}$$

PDE solution via Laplace Transform in time domain (constant flow F, zero initial conditions)

Let us consider the PDE:

$$\frac{\partial T}{\partial t} = -v \frac{\partial T}{\partial x} - aT + b\overline{Q}$$

where $a = \frac{\overline{\kappa}}{S\rho C_e}$, $b = \frac{1}{S\rho C_e}$, $v = \frac{F}{S}$, in order to streamline the notation.

If we carry out Laplace transform of the "temporal" component of $T(x,t)$, we will denote such transform as $\mathbb{T}(x,s)$. It can be proved that $\mathscr{L}\left(\frac{\partial}{\partial x}T\right) = \frac{\partial}{\partial x}\mathscr{L}(T)$, so:

$$s\mathbb{T}(x,s) - T(x,0) = -v\frac{\partial}{\partial x}\mathbb{T}(x,s) - a\mathbb{T}(x,s) + b\overline{\mathbb{Q}}(x,s)$$

and arranging things a bit, we have:

$$v\frac{\partial}{\partial x}\mathbb{T}(x,s) + (s+a)\mathbb{T}(x,s) = T(x,0) + b\overline{\mathbb{Q}}(x,s)$$

The important fact to notice is that this is a 1st-order ODE on the variable x, if we consider s, the Laplace variable, as a "numeric variable"; indeed, Laplace transform allows us to operate "algebraically" with "s" as we would do with any "number" or "constant parameter".

We will set, for simplicity, the initial condition $T(x,0) = 0$, as, indeed, once the heat exchanger gets "flushed" such an initial condition will not be so important... and, well, we want just transfer functions as the goal of our modelling.

We will also set heating power constant throughout all the exchanger's length, so we get the ODE:

$$\frac{\partial}{\partial x}\mathbb{T}(x,s) + (s+a)/v \cdot \mathbb{T}(x,s) = b/v \cdot \overline{\mathbb{Q}}(s)$$

Recall now that the solution of, say, $\dot{T} + pT = h$ with constant p and h is $T(x) = e^{-px} \cdot T(0) + (1 - e^{-px})\frac{h}{p}$, (obtained with Laplace or with any other textbook formula). Hence, the solution to the above ODE is:

$$\mathbb{T}(x,s) = (1 - e^{-\frac{s+a}{v}\cdot x})\frac{b}{(s+a)} \cdot \bar{\mathbb{Q}}(s) + \mathbb{T}(0,s)e^{-\frac{s+a}{v}\cdot x}$$

But $\mathbb{T}(0,s) = \mathbb{T}_{in}(s)$, so we can rewrite as:

$$\mathbb{T}(x,s) = (1 - e^{-\frac{s+a}{v}\cdot x})\frac{b}{(s+a)} \cdot \bar{\mathbb{Q}}(s) + \mathbb{T}_{in}(s)e^{-\frac{s+a}{v}\cdot x}$$

In this way, we have obtained the transfer function matrix for any x for the temperature at said position with respect to heat power and inlet temperature.

Note that $e^{-\frac{s+a}{v}\cdot x}$ can be factorised as $e^{-\frac{s+a}{v}\cdot x} = e^{-\frac{s}{v}\cdot x} e^{-\frac{a}{v}\cdot x}$, which can be interpreted as a delay of x/v time units ($e^{-\frac{s}{v}\cdot x}$) which multiplies an exponential decay as length increases ($e^{-\frac{a}{v}\cdot x}$) .

Steady state (thermal equilibrium): if heating power Q were constant and so it were inlet temperature T_{in} then, when any transient effect has ended, the temperature profile will be, substituting s by zero (as in dc-gain computations):

$T_{eq}(x) = (1 - e^{-\frac{a}{v}\cdot x})\frac{b}{a} \cdot \bar{Q}_{eq} + T_{in,eq}e^{-\frac{a}{v}\cdot x}$, being coincident with the stationary solution recalled above, of course.

Transient behaviour of the outlet temperature:

Usually, the only temperature of interest is at $x = L$ (the heat exchanger's outlet temperature), so we have:

$$\mathbb{T}_{out}(s) = \mathbb{T}(L,s) = (1 - e^{-\frac{L}{v}\cdot s} e^{-\frac{aL}{v}})\frac{b}{s+a} \cdot \bar{\mathbb{Q}}(s) + \mathbb{T}_{in}(s)e^{-\frac{L}{v}\cdot s} e^{-\frac{aL}{v}}$$

We see that there is a transport delay $e^{-\frac{L}{v}\cdot s}$, coincident with the "flushing time", as expected... Denoting as $\phi = L/v$ such delay, we have:

$$\mathbb{T}_{out}(s) = (1 - e^{-\phi\cdot s}e^{-a\phi})\frac{b}{s+a} \cdot \bar{\mathbb{Q}}(s) + e^{-\phi\cdot s}e^{-a\phi} \cdot \mathbb{T}_{in}(s)$$

se we can write the transfer function matrix as follows:

$$\mathbb{T}_{out}(s) = \begin{pmatrix} (1 - e^{-\phi \cdot s} e^{-a\phi}) \frac{b}{s+a} & e^{-\phi \cdot s} e^{-a\phi} \end{pmatrix} \cdot \begin{pmatrix} \overline{\mathbb{Q}}(s) \\ \mathbb{T}_{in}(s) \end{pmatrix}$$

Recall that physical parameters are $a = \frac{\overline{\kappa}}{S\rho C_e}$, $b = \frac{1}{S\rho C_e}$, $v = \frac{F}{S}$, $\phi = \frac{L}{v} = \frac{V}{F}$, thus

$a\phi = \frac{\overline{\kappa}L}{F\rho C_e} = \frac{\overline{\kappa}L}{vS\rho C_e}$

*Remember that time variations of input flow would make the above derivations invalid, as there is nonlinearity due to the product $F\frac{\partial T}{\partial x}$, so it does not appear in the Laplace transform's input vector. Of course, we may approximate with a linearization of the PDE as $\frac{\partial T}{\partial t} = -\frac{1}{S}F_0\frac{\partial T}{\partial x} + -\frac{1}{S}F\frac{\partial T_0(x)}{\partial x} - \frac{\overline{\kappa}}{S\rho C_e}T + \frac{\overline{Q}}{S\rho C_e}$, knowing $T_0(x)$, computing a stationary profile for a nominal flow F_0, and then a transfer function with respect to "small" flow increments can be computed in a similar way as we carried out above. I leave that as an exercise for you.