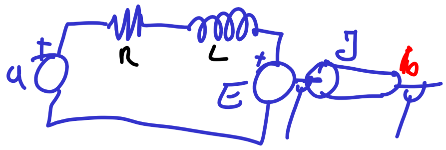
Motor + gears modelling

Antonio Sala

Presentations in video:
`https://personales.upv.es/asala/YT/V/motccmodelEN.html` (no gears)
`https://personales.upv.es/asala/YT/V/motccmodel2EN.html` (no inductance)
`https://personales.upv.es/asala/YT/V/motengEN.html` (with gearbox)

1 Electric motor 1 axis, no gearbox



Input variable is u; intended application: 'speed' control (angular position is not relevant). SIX equations and six unknowns (V_R, V_L, E, T, i, ω):

$$u = V_R + V_L + E \tag{1}$$

$$V_R = R \cdot I \tag{2}$$

$$\frac{dI}{dt} = \frac{1}{L} V_L \tag{3}$$

$$E = k \cdot \omega \tag{4}$$

$$T = k \cdot I \tag{5}$$

$$\frac{d\omega}{dt} = \frac{1}{J}(T - b\omega) \tag{6}$$

Nota: In a 'position control' application, the angular position should be incorporated into the model by adding equation $\frac{d\theta}{dt} = \omega$ (one more equation and one more unknown, the model would still be correct). We will not do that here for brevity. Formally, the state variable θ is said not to be *observable* if we only have angular velocity and ammeter measurements.

1.1 Internal representation (normalised state equation)

We must eliminate V_R, V_L, E, T and leave everything in terms of input 'u' and state variables (I, ω). Substituting equations (1,2,4) in (3) and substituting (5) in (6), we get:

$$\frac{dI}{dt} = \frac{1}{L}(u - RI - k\omega) \tag{7}$$

$$\frac{d\omega}{dt} = \frac{1}{J}(kI - b\omega) \tag{8}$$

Matrix form $x = [I; \omega]$:

$$\frac{dx}{dt} = \begin{pmatrix} -R/L & -k/L \\ +k/J & -b/J \end{pmatrix} x + \begin{pmatrix} 1/L \\ 0 \end{pmatrix} u \tag{9}$$

Output equations: The variable or variables that might interest the user depends on each application. If, for example, the user were interested in the torque and the current in the resistor $y = [T; I]$, we would add:

$$y = \begin{pmatrix} k & 0 \\ 1 & 0 \end{pmatrix} x + \begin{pmatrix} 0 \\ 0 \end{pmatrix} u \tag{10}$$

in order to indicate to subsequent theoretical developments or to simulation software that they must calculate and represent these outputs once it solves (9) which is where the physics of the energy exchange between the system elements is.

1.2 Reduced-order model, for small L

If $L \to 0$ then the electrical equation ends in $0 = L\frac{dI}{dt} = V_L$, so we change L by a short-circuit. Instead of redoing again everything, we can set to zero the left-hand side of the first state equation. So, we get $u - RI - k\omega = 0$, i.e.,

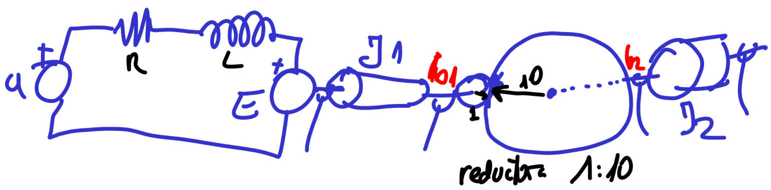
$$I = \frac{1}{R} \cdot (u - k\omega_1)$$

and, replacing it in the 2nd state equation, we would get a 1st-order approximation:

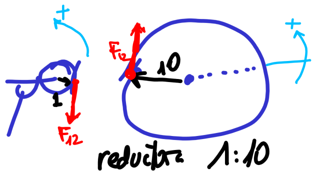
$$\frac{d\omega}{dt} = \frac{-1}{J} \cdot \left(\frac{k^2}{R} + b\right) \cdot \omega_1 + \frac{k}{RJ} u \tag{11}$$

so the resistor acts as a sort of 'increased friction' when viewed from the 'mechanical' part ω.

2 Electric motor with 2 axes connected by a 10:1 reducer gearbox



We would add the dynamics of axis 2, the interaction force between the axes F_{12} and the transmission ratio 1:10 (see the figure below to understand the meaning of F_{12}).



Note: I'm going to start with the previous state equations [(7)–(8)] that summarize 6 equations in 2, for simplicity (it's not mandatory to do so; obviously, you could model by adapting [(1)–(6)]). This assumes that the four variables we elliminated (V_L, V_R, E, T) are not of interest to me. If that were not the case, I should keep their equations (or recover them when writing output equations later on).

We will therefore start from:

$$\frac{dI}{dt} = \frac{1}{L}(u - RI - k\omega_1) \tag{12}$$

$$\frac{d\omega_1}{dt} = \frac{1}{J_1}(kI - b_1\omega_1 - F_{12} \cdot 1) \tag{13}$$

where, since there are two angular velocities, I have changed ω in (8) to ω_1 above. To these equations, we add:

$$\frac{d\omega_2}{dt} = \frac{1}{J_2}(-b_2\omega_2 - F_{12} \cdot 10) \tag{14}$$

$$\omega_2 = -\frac{1}{10}\omega_1 \tag{15}$$

Where the torques are 10 times larger on axis 2 ($F_{12} \cdot 10$) but the angular velocities are 10 times smaller. This new model [(12)–(15)] of 4 equations and 4 unknowns $(I, \omega_1, \omega_2, F_{12})$ is now complete (but it is not 'normalized').

Signs: we assume positive sign as 'counterclockwise' rotation. The signs are correct because: (a) the frictions go in the opposite direction of the speed, (b) the axes rotate in opposite directions and, (c) when $F_{12} > 0$, it pushes axis 1 in the 'negative' direction it also does so on axis 2, as depicted in the figure above.

2.1 Normalized state equations

Due to (15), the two angular velocities are not 'independent variables': they have a 'kinematic link' and in fact there is only ONE 'mechanical degree of freedom': although conceptually both ω_1 and ω_2 are 'states' (they store kinetic energy), there is a representation of the mathematical model where only one of them appears. Indeed, we can eliminate ω_2 since it is a linear function of ω_1 (or eliminate ω_1, depending on whether we want a 'motor-side' or 'load-side' reduction of the model). In linear systems theory, doing this is often called 'obtaining a *minimal realization*'.

I have decided to eliminate ω_2: I am looking for the equivalent 'motor-side' dynamics, that is, the dynamics of ω_1.

First, the normalized internal representation will for sure need to elliminate F_{12}. We'll do it by using (15) to rewrite (14) as:

$$\frac{d}{dt}(-\omega_1/10) = \frac{1}{J_2}(-b_2(-\omega_1/10) - F_{12} \cdot 10) \tag{16}$$

We will call it $N = 10$, and write the equations in terms of N to make it valid for any gear ratio. Instead of using (16) for our specific model (which would be perfectly valid), we will use the letter N and substitute $\omega_2 = -\omega_1/N$ in (14) so that we will have:

$$\frac{d}{dt}(-\omega_1/N) = \frac{1}{J_2}(-b_2 \cdot (-\omega_1/N) - F_{12} \cdot N) \tag{17}$$

That is, multiplying by $-N$ both sides of the equation:

$$\frac{d\omega_1}{dt} = \frac{1}{J_2}(-b_2\omega_1 + F_{12} \cdot N^2) \tag{18}$$

So that, equating with the right side of (13) we get:

$$\frac{1}{J_1}(kI - b_1\omega_1 - F_{12} \cdot 1) = \frac{1}{J_2}(-b_2\omega_1 + F_{12} \cdot N^2) \tag{19}$$

and, solving for F_{12} and operating, we get:

$$F_{12} = \frac{1}{J_1 N^2 + J_2}\left(J_1 b_2 \omega_1 - J_2 b_1 \omega_1 + J_2 k i\right) \tag{20}$$

and substituting in (13), we get, after doing some operations to make it 'pretty':

$$\frac{d\omega_1}{dt} = \frac{1}{J_1 + J_2/N^2}\left(kI - b_1\omega_1 - \frac{b_2}{N^2}\omega_1\right) \tag{21}$$

Hence, the final normalized state equations are:

$$\frac{dI}{dt} = \frac{1}{L}\left(u - RI - k\omega_1\right) \tag{22}$$

$$\frac{d\omega_1}{dt} = \frac{1}{J_1 + J_2/N^2}\left(kI - (b_1 + \frac{b_2}{N^2})\omega_1\right) \tag{23}$$

being $J_{eq} := J_1 + J_2 \cdot \frac{1}{N^2}$ named *equivalent moment of inertia* and $b_{eq} = b_1 + \frac{b_2}{N^2}$ denoted as equivalent friction coefficient.

Nota: the result is coincident with the 'rule' of mechanical engineering that the inertias on the other side of the gearbox are divided by N^2, and the torques are divided by N (in effect, the friction on axis 2, $F_{friction2} = b_2\omega_2$, ends up being in 'equivalent terms on axis 1' as $F_{friction2}/N = b_2\omega_2/N = b_2\omega_1/N^2$, signs aside. Therefore, if N is large, the reduction gear basically 'isolates' the motor from the rest of the world and the equivalent inertia and friction are almost equal to J_1 and b_1, respectively: the motor behaviour will be almost independent of the load on the second axis if N is large enough.

In matrix form, we would have:

$$\frac{d}{dt}\begin{pmatrix} I \\ \omega_1 \end{pmatrix} = \begin{pmatrix} -R/L & -k/L \\ +k/J_{eq} & -b_{eq}/J_{eq} \end{pmatrix} \cdot \begin{pmatrix} I \\ \omega_1 \end{pmatrix} + \begin{pmatrix} 1/L \\ 0 \end{pmatrix} \cdot u \tag{24}$$

Note that we would need to complement with suitable *output* equations, of course (details omitted for brevity).

2.2 Load torque (disturbance)

IF I added a load torque over axis 1, named $T_{load,1}$, or over axis 2, say T_{load2}, reducing to axis 1 we would get:

$$\frac{d\omega_1}{dt} = \frac{1}{J_1}(kI - b_1\omega_1 - F_{12} + T_{load,1}) \tag{25}$$

$$\frac{d\omega_2}{dt} = \frac{1}{J_2}(-b_2\omega_2 - F_{12} \cdot N + T_{load,2}) \tag{26}$$

So, replacing $\omega_2 = \omega_1/N$ and multiplying times N both sides of the latter equation, we get:

$$\frac{d\omega_1}{dt} = \frac{1}{J_2}(-b_2\omega_1 - F_{12} \cdot N^2 + T_{load,2} \cdot N) \tag{27}$$

Hence, equation $\dot{\omega}_1$ results in:

$$\frac{1}{J_1}(kI - b_1\omega_1 - F_{12} + T_{load,1}) = \frac{1}{J_2}(-b_2\omega_1 + F_{12} \cdot N^2 + T_{load,2} \cdot N) \tag{28}$$

Thus, solving for F_{12} yields:

$$F_{12} = \frac{J_1 b_2 \omega_1 - J_1 T_{load,2} N + J_2 kI - J_2 b_1 \omega_1 + J_2 T_{load,1}}{J_2 + J_1 N^2} \tag{29}$$

so, last, we substitute in (25) getting:

$$\frac{d\omega_1}{dt} = \frac{1}{J_1 + \frac{J_2}{N^2}} \cdot \left(kI - (b_1 + \frac{b_2\omega_1}{N^2})\omega_1 + T_{load,1} + \frac{T_{load,2}}{N} \right) \tag{30}$$

In matrix form, we would have:

$$\frac{d}{dt}\begin{pmatrix} I \\ \omega_1 \end{pmatrix} = \begin{pmatrix} -R/L & -k/L \\ +k/J_{eq} & -b_{eq}/J_{eq} \end{pmatrix} \cdot \begin{pmatrix} I \\ \omega_1 \end{pmatrix} + \begin{pmatrix} 1/L & 0 \\ 0 & \frac{1}{J_{eq}} \end{pmatrix} \cdot \begin{pmatrix} u \\ T_{load,eq} \end{pmatrix} \tag{31}$$

where $T_{load,eq} := T_{load,1} + \frac{1}{N} T_{load,2}$.