Analysis of damped vibration modes in free response of a 3-mass, 4-spring system

This code was run in Matlab **R2022a**

Presentation in video: http://personales.upv.es/asala/YT/V/moll3freeEN.html

Objective: analyzing the properties of a system of 3 masses and 4 springs, specifically its "damped vibration modes" in the ***free response***. They will be similar, but not the same, as the "forced" modes (resonant frequencies), which will be discussed in other materials.

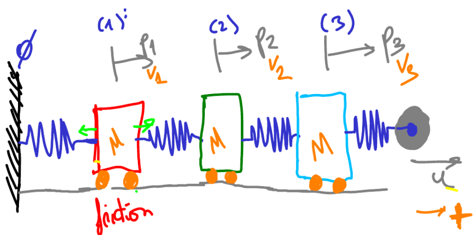
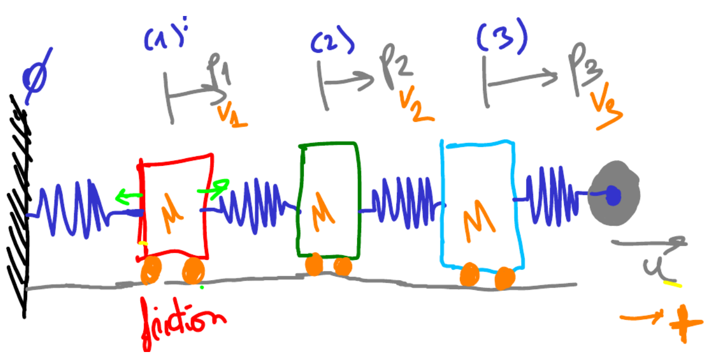


Table of Contents

System Model........1
Free response (non-zero i.c.): modes and properties........2
Theory in a nutshell........2
Numerical computation........3
MODE 1:........4
MODE 2........7
MODE 3........10

System Model



The input signal is "displacement of the rightmost end", we intentionally chose it that way. Other videos in the collection discuss modeling in detail.

```
k=1; b=0.1; M=1; %constant parameters

A=[0 1 0 0 0 0; ...
    -2*k/M -b/M k/M 0 0 0;...
    0 0 0 1 0 0;...
    k/M 0 -2*k/M -b/M k/M 0;...
    0 0 0 0 0 1; ...
    0 0 k/M 0 -2*k/M -b/M];

B=[0;0;0;0;0;k/M];
```

Outputs will be the three positions:

```
C=[ 1 0 0 0 0 0; ...
    0 0 1 0 0 0; ...
    0 0 0 0 1 0];
```

We create a Control System Toolbox LTI system with:

```
sys=ss(A,B,C,0);
```

We may obtain transfer functions (but we will not use them in this video, as we are dealing with analysing the "free" response, i.e., *zero* input so B is not relevant)

```
zpk(sys) %relative degree 6, 4, 2 with respect to input
```

```
ans =

  From input to output...
                                   1
   1:  -----------------------------------------------------------
       (s^2 + 0.1s + 0.5858) (s^2 + 0.1s + 2) (s^2 + 0.1s + 3.414)

                            (s^2 + 0.1s + 2)
   2:  -----------------------------------------------------------
       (s^2 + 0.1s + 0.5858) (s^2 + 0.1s + 2) (s^2 + 0.1s + 3.414)

                    (s^2 + 0.1s + 1) (s^2 + 0.1s + 3)
   3:  -----------------------------------------------------------
       (s^2 + 0.1s + 0.5858) (s^2 + 0.1s + 2) (s^2 + 0.1s + 3.414)

Continuous-time zero/pole/gain model.
```

Free response (non-zero i.c.): modes and properties

Theory in a nutshell

The response to arbitrary initial conditions is a superposition of different independent modes, obtained by diagonalizing A.

Consider $A = VDV^{-1}$, with D diagonal (i.e., distinct non-repeated eigenvalues λ_i in the diagonal). For the purists out there, Jordan blocks should be added to this... Not done.

Change variable with $\xi = V^{-1}x$, i.e., $V\xi = x$.

Then $\frac{d\xi}{dt} = V^{-1}\frac{dx}{dt} = V^{-1}Ax = V^{-1}(VDV^{-1})V\xi = D\xi$

So, the dynamics of ξ is diagonal $\frac{d\xi_i}{dt} = \lambda_i\xi_i$, with exponential solution $\xi_i(t) = \xi_i(0)e^{\lambda_i t}$.

The initial conditions $\xi_i(0) = 1$, $\xi_j(0) = 0$ for $j \neq i$, exciting a single "mode", end up as $x(0) = V_{[i]}$, being $V_{[i]}$ the i-th column of V, i.e. the i-th eigenvector.

Numerical computation

```
[V,D]=eig(A);
V
```

```
V = 6×6 complex
   0.0064 + 0.2379i   0.0064 - 0.2379i  -0.0144 - 0.4080i  -0.0144 + 0.4080i ⋯
  -0.4397 + 0.0000i  -0.4397 - 0.0000i   0.5774 + 0.0000i   0.5774 + 0.0000i
  -0.0091 - 0.3364i  -0.0091 + 0.3364i  -0.0000 + 0.0000i  -0.0000 - 0.0000i
   0.6219 + 0.0000i   0.6219 + 0.0000i  -0.0000 - 0.0000i  -0.0000 + 0.0000i
   0.0064 + 0.2379i   0.0064 - 0.2379i   0.0144 + 0.4080i   0.0144 - 0.4080i
  -0.4397 - 0.0000i  -0.4397 + 0.0000i  -0.5774 + 0.0000i  -0.5774 - 0.0000i
```

```
DD = diag(D)' %complex eigenvalues: exponential*sin/cos
```

```
DD = 1×6 complex
  -0.0500 - 1.8471i  -0.0500 + 1.8471i  -0.0500 - 1.4133i  -0.0500 + 1.4133i ⋯
```

Eigenvalues and eigenvectors come in complex conjugate pairs.

All modes have the same "decay rate" (real part).

```
-pi./real(DD) %approx 5% settling time
```

```
ans = 1×6
   62.8319   62.8319   62.8319   62.8319   62.8319   62.8319
```

We'll sort them from lowest to highest frequency (imaginary part):

```
frq=abs(imag(DD([5 3 1]))) %frecuencias "propias", ordenadas
```

```
frq = 1×3
    0.7637    1.4133    1.8471
```

Period of the oscillations at each frequency is:

```
prd=2*pi./frq
```

```
prd = 1×3
    8.2270    4.4457    3.4017
```

MODE 1:

```
mode_eigenvalue=DD(5)
```

```
mode_eigenvalue = -0.0500 - 0.7637i
```

```
mode_eigenvalue=DD(6) %complex conjugate
```

```
mode_eigenvalue = -0.0500 + 0.7637i
```

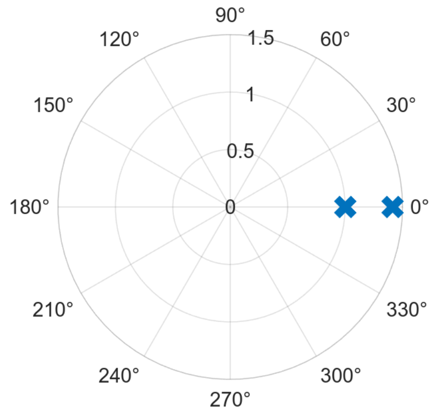
```
mode1positions=V([1 3 5],5) %positions are states 1, 3, 5
```

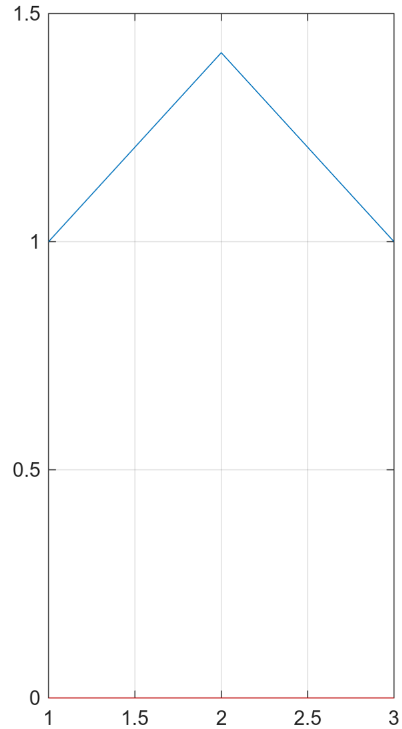
```
mode1positions = 3×1 complex
   0.3971 - 0.0000i
   0.5615 + 0.0000i
   0.3971 + 0.0000i
```

```
mode_1=mode1positions/mode1positions(1) %scaling/rotating so mass 1 displaces 1 meter
```

```
mode_1 = 3×1 complex
   1.0000 + 0.0000i
   1.4142 + 0.0000i
   1.0000 + 0.0000i
```

```
figure()
subplot(1,2,1)
polarplot(mode_1,'x',MarkerSize=12,LineWidth=4)
subplot(1,2,2)
plot(real(mode_1)),yline(0,'r'), grid on,
```



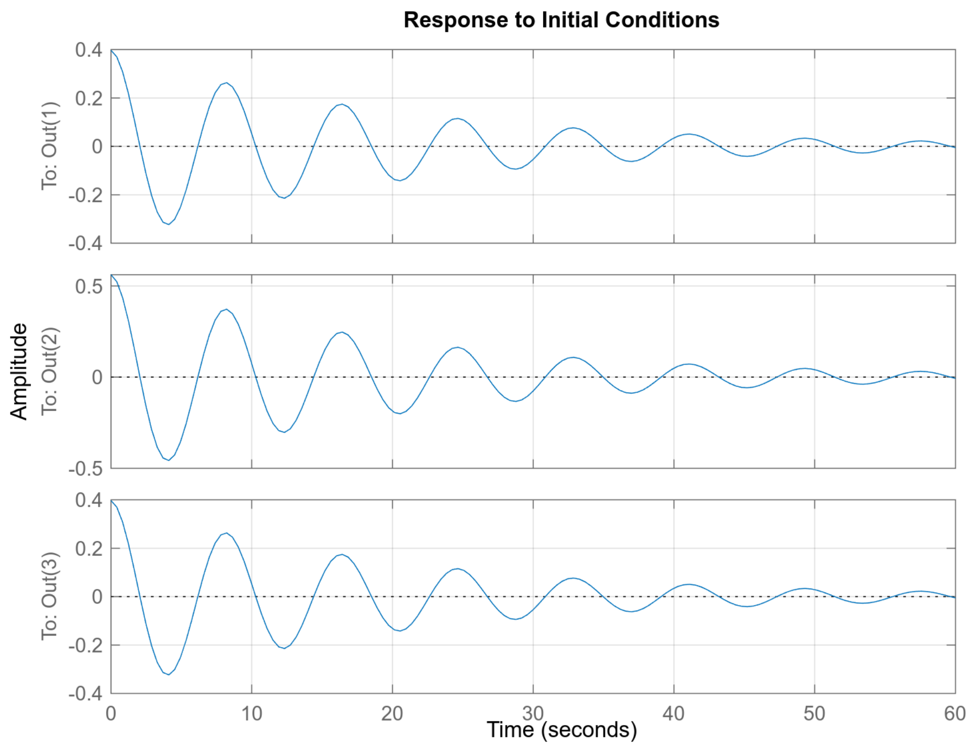


Time response:

```
figure()
x0=real(V(:,5))
```

```
x0 = 6×1
    0.3971
   -0.0199
    0.5615
   -0.0281
    0.3971
   -0.0199
```

```
initial(sys,x0,60), grid on
```

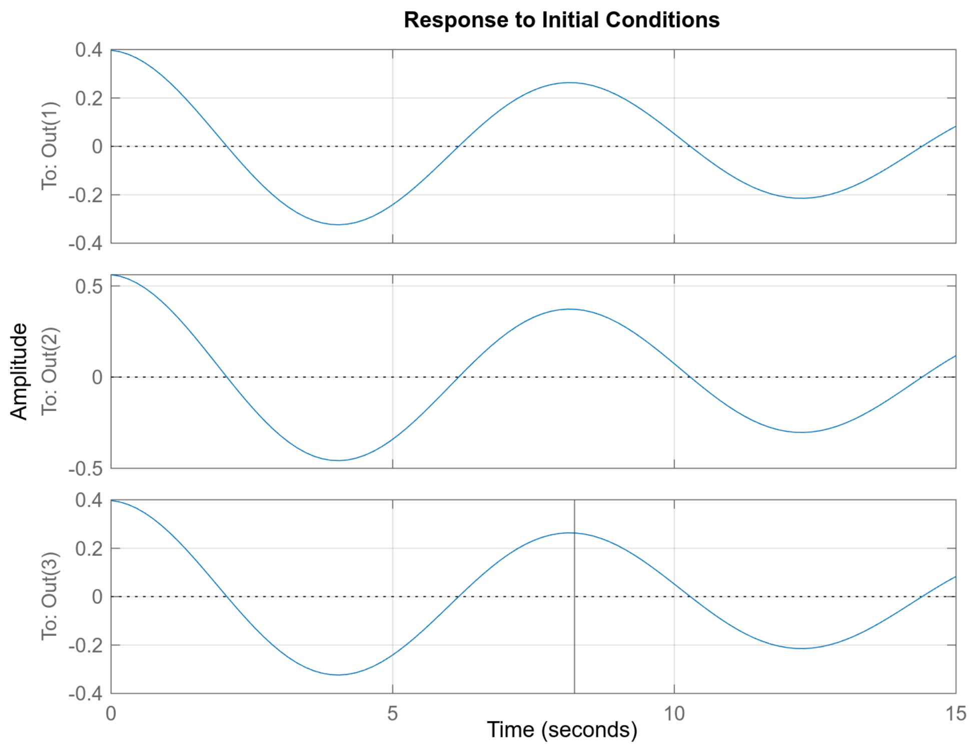


Let's check oscillation period:

```
prd(1)
```

```
ans = 8.2270
```

```
initial(sys,x0,15), grid on, xline(prd(1))
```



MODE 2

```
figure()
mode_eigenvalue=DD(3)
```

```
mode_eigenvalue = -0.0500 - 1.4133i
```

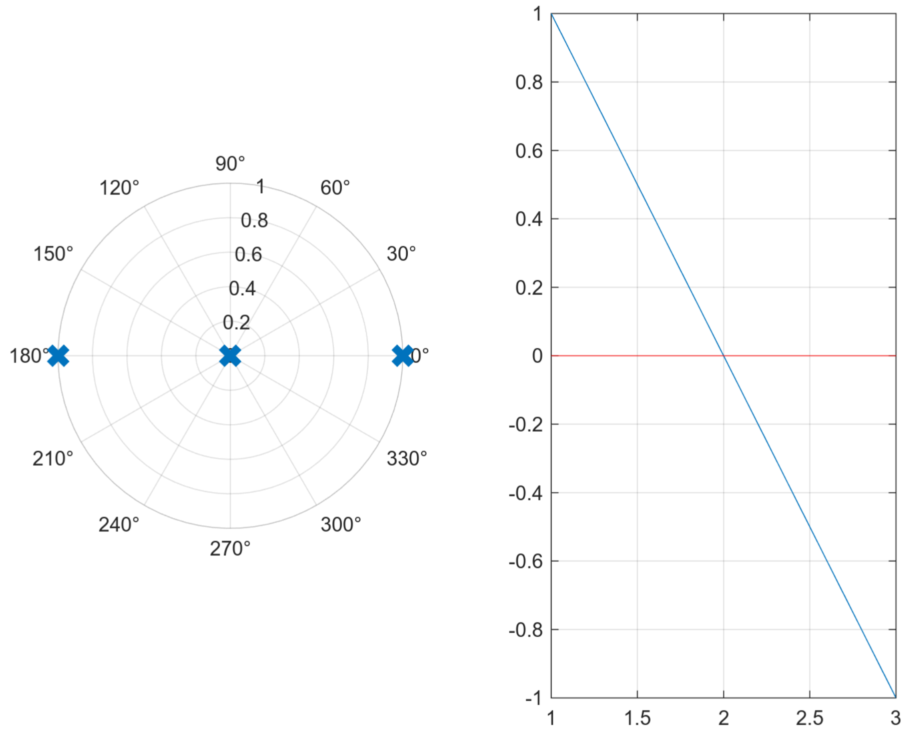
```
mode2positions=V([1 3 5],3) %sólo posición
```

```
mode2positions = 3×1 complex
  -0.0144 - 0.4080i
  -0.0000 + 0.0000i
   0.0144 + 0.4080i
```

```
mode2positions=mode2positions/mode2positions(1)
```

```
mode2positions = 3×1 complex
   1.0000 + 0.0000i
  -0.0000 - 0.0000i
  -1.0000 + 0.0000i
```

```
subplot(1,2,1)
polarplot(mode2positions,'x',MarkerSize=12,LineWidth=4)
subplot(1,2,2)
plot(real(mode2positions)),yline(0,'r'), grid on
```

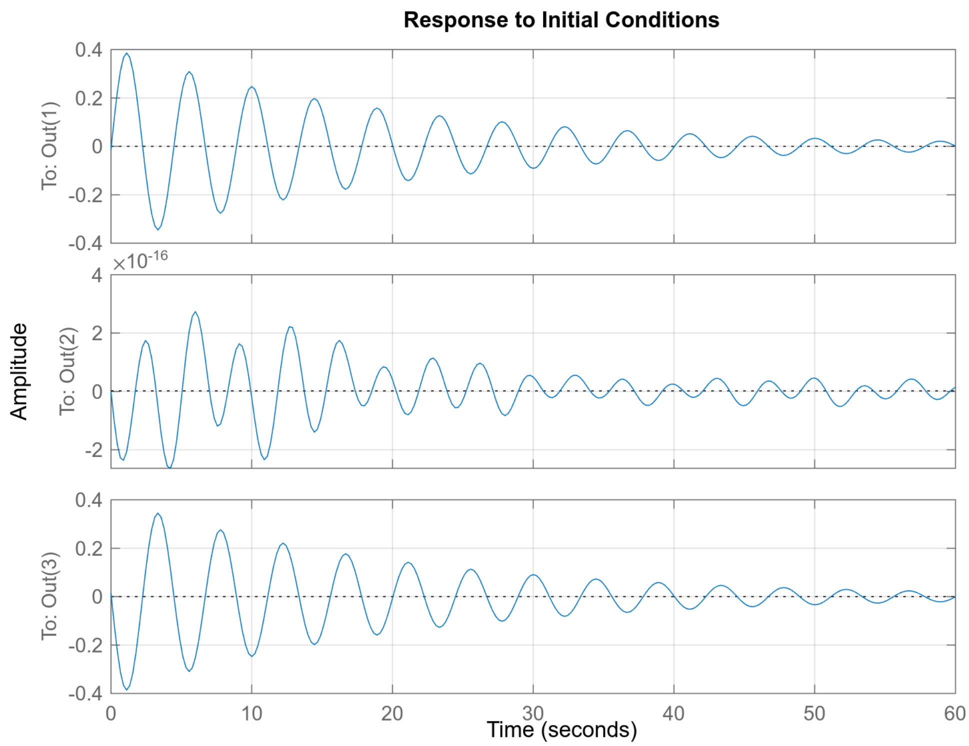


Time response

```
figure()
x0=real(V(:,3))
```

```
x0 = 6×1
   -0.0144
    0.5774
   -0.0000
   -0.0000
    0.0144
   -0.5774
```

```
initial(sys,x0,60), grid on
```

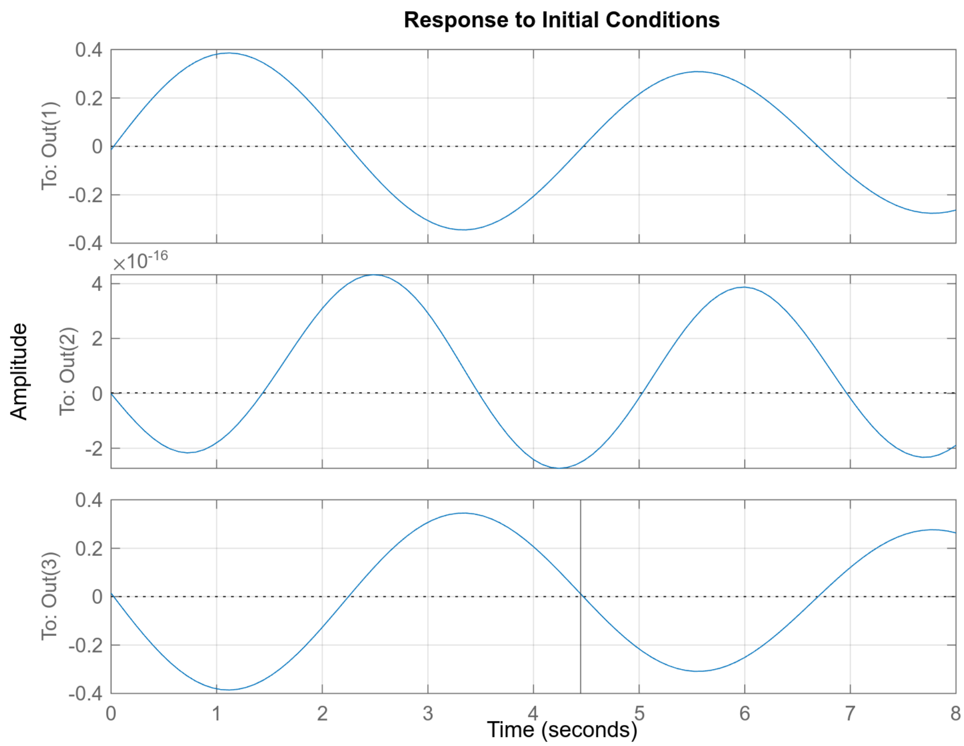


Oscillation period:

```
prd(2)
```

```
ans = 4.4457
```

```
initial(sys,x0,8), grid on, xline(prd(2))
```



MODE 3

```
figure()
mode_eigenvalue=DD(1)
```

```
mode_eigenvalue = -0.0500 - 1.8471i
```

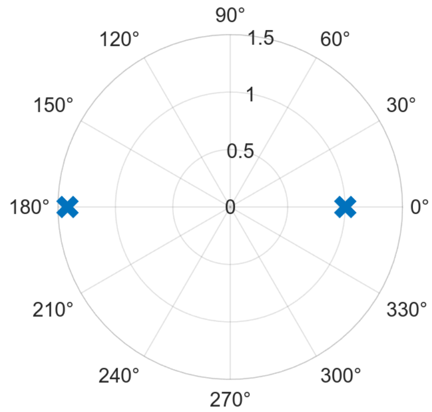
```
mode3positions=V([1 3 5],1) %sólo posición
```

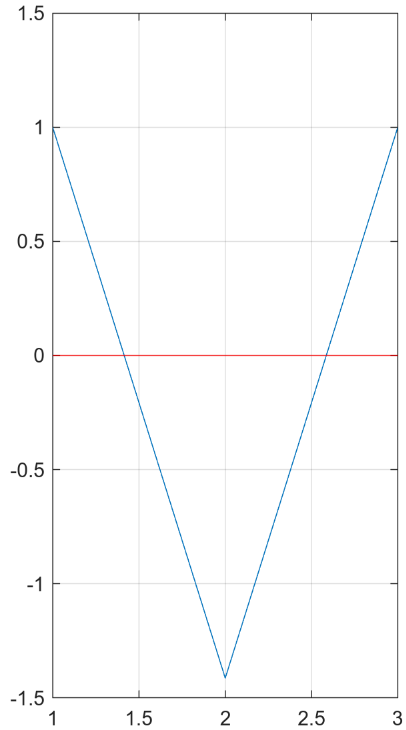
```
mode3positions = 3×1 complex
   0.0064 + 0.2379i
  -0.0091 - 0.3364i
   0.0064 + 0.2379i
```

```
mode3positions=mode3positions/mode3positions(1)
```

```
mode3positions = 3×1 complex
   1.0000 + 0.0000i
  -1.4142 - 0.0000i
   1.0000 + 0.0000i
```

```
subplot(1,2,1)
polarplot(mode3positions,'x',MarkerSize=12,LineWidth=4)
subplot(1,2,2)
plot(real(mode3positions)),yline(0,'r'), grid on
```

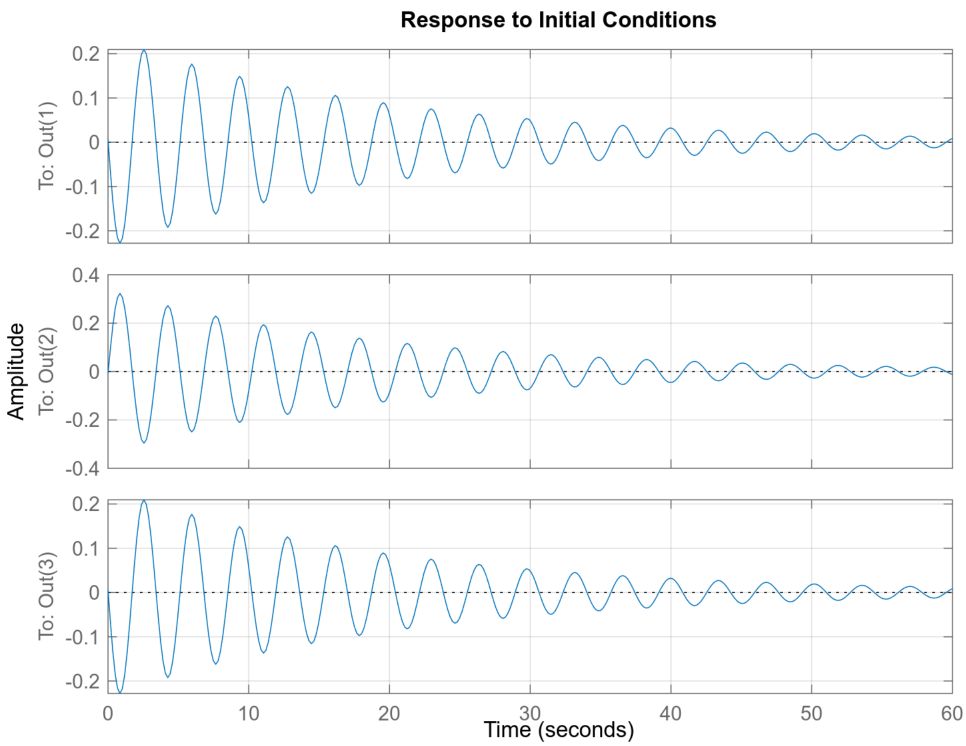





```
figure()
x0=real(V(:,1))
```

```
x0 = 6×1
    0.0064
   -0.4397
   -0.0091
    0.6219
    0.0064
   -0.4397
```

```
initial(sys,x0,60), grid on
```

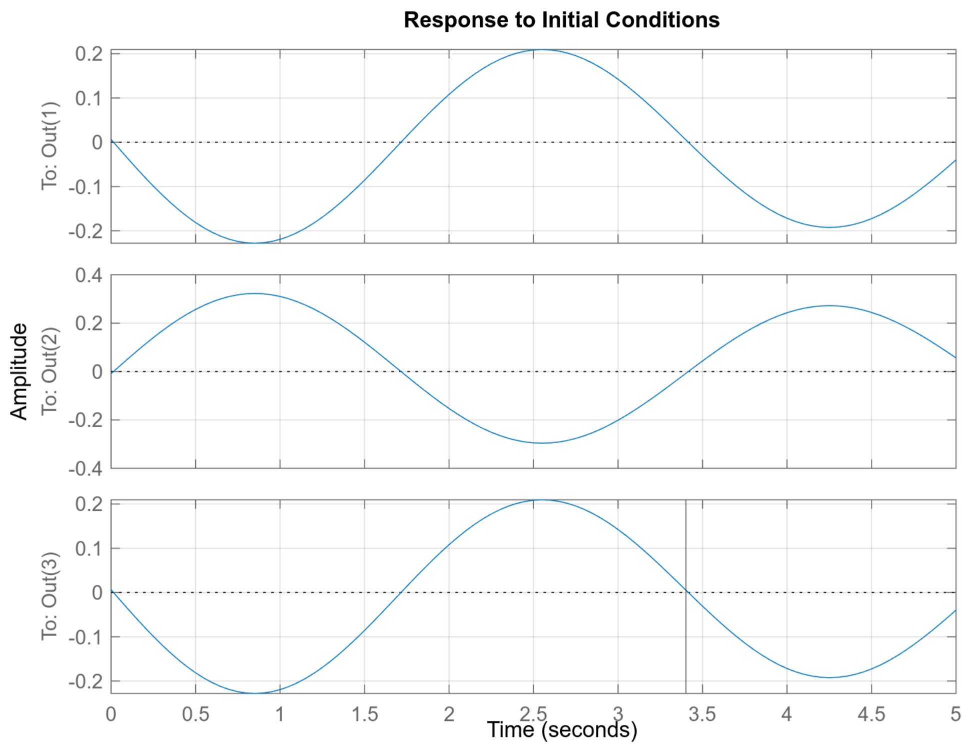


Oscillation period:

```
prd(3)
```

```
ans = 3.4017
```

```
initial(sys,x0,5), grid on, xline(prd(3))
```


Response to Initial Conditions
Amplitude
To: Out(1)
To: Out(2)
To: Out(3)
0.2
0.1
0
-0.1
-0.2
0.4
0.2
0
-0.2
-0.4
0.2
0.1
0
-0.1
-0.2
0
0.5
1
1.5
2
2.5
3
3.5
4
4.5
5
Time (seconds)