Longitudinal Aircraft Dynamics: "phugoid" mode, simplified order 2 Equilibrium points, linearization, stability

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Lecture Notes on modeling and Control of Complex Systems

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Link to materials, comments, etc. at description and personal website

Video-presentación disponible en:

http://personales.upv.es/asala/YT/V/**fugeqlinEN**.html

Outline

Motivation:
Under certain approximations, the dynamics of an aircraft can approximate to a "point mass" mode called "phugoid".

Objectives:
Analyze the equilibrium points and the stability of said dynamics.

Contents:
Review of prior concepts. Equilibrium points. Linearization. Stability. Conclusions.

Longitudinal phugoid mode assumptions

- Planar (x, y) movement, longitudinal displacement vs. elevation.
- Negligible effect of Moment of Inertia. The aerodynamics aligns it "infinitely fast" at a fixed angle α to the airspeed (the plane behaves like a weathervane).
- Negligible aircraft length with respect to trajectory curvature radius.

In other words, a 2GL dynamic is assumed, where the angle depends on the trajectory (x, y), order 4 in positions, order 2 in speeds.

***Formulae from Zhukovski (1891)-Lanchester (1908).**
https://en.wikipedia.org/wiki/Nikolay_Zhukovsky_(scientist)

Phugoid mode equations (normalized internal repr.)

Replacing $Lift = l \cdot v^2$ and $Drag = d \cdot v^2$ with expressions that, for simplicity, we only make dependent on v^2 (actually, also dependent on the orientation of the aircraft with respect to airspeed velocity vector, ...), we have:

$$\frac{dv}{dt} = -g \sin\theta - \frac{d}{m} \cdot v^2 + \frac{1}{m} u$$
$$\frac{d\theta}{dt} = -g \frac{\cos\theta}{v} + \frac{l}{m} \cdot v$$

With this state equation, we can now carry out simulations, e.g. with Matlab `ode45`.

*We will NOT consider the possibility of airspeed-dependen thrust, ie.e., of $u(v)$ in later linearizations, but it might be relevant in some cases (in particular in propeller engines).

Equilibrium conditions

Pitch and airspeed will keep constant if derivatives are zero:

$$0 = -g \sin \theta_{eq} - \frac{d}{m} \cdot v_{eq}^2 + \frac{1}{m} u_{eq}$$

$$0 = -g \frac{\cos \theta_{eq}}{v_{eq}} + \frac{l}{m} \cdot v_{eq}$$

Operating:

$$mg \sin \theta_{eq} = -d \cdot v_{eq}^2 + u_{eq}$$

$$mg \cos \theta_{eq} = +l \cdot v_{eq}^2$$

Two equations, three unknowns: if we fix one variable we can solve for the other two.

[1] *glider $u = 0$, descent trajectory:

$$\theta_{eq} = \arctan \frac{-d}{l} \qquad v_{eq}^2 = \frac{mg}{\sqrt{d^2 + l^2}} \qquad \mathit{Lift}^2 + \mathit{Drag}^2 = \mathit{Weight}^2$$

[2] Given the desired equilibrium pitch angle:

$$v_{eq} = \sqrt{\frac{mg}{l} \cos \theta_{eq}}$$

$$u_{eq} = mg \left(\sin \theta_{eq} + \frac{d}{l} \cos \theta_{eq} \right)$$

Level flight: $\theta_{eq} = 0$, $v_{eq} = \sqrt{mg/l}$, $u_{eq} = mg \cdot d/l$.

Flaps: incrementing "l" achieves equilibrium at lower airspeeed. Thrust depends on whether d/l chanves or not.

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[3] Given thrust, force balance is $L^2 + (u - D)^2 = m^2 g^2$, hence:

$$l^2 v_{eq}^4 + (u_{eq} - d v_{eq}^2)^2 = m^2 g^2$$

so

$$(d^2 + l^2) v_{eq}^4 - 2 d u_{eq} v_{eq}^2 + (u_{eq}^2 - m^2 g^2) = 0$$

is a biquadratic equation to solve for v_{eq}^2 given u_{eq}.

$$v_{eq}^2 = \frac{d u_{eq} \pm \sqrt{(d^2 + l^2) m^2 g^2 - l^2 u_{eq}^2}}{d^2 + l^2}$$

*If $mg < u \leq mg \sqrt{1 + \left(\frac{d}{l}\right)^2}$ there are two positive solutions, and two angles.

*If $u > mg \frac{\sqrt{l^2 + d^2}}{l}$ there are no real solutions, so phugoid does not reach equilibrium.

Stability: linearized model

*Incremental coords.:

$$\frac{dv}{dt} = -\frac{2d \cdot v_{eq}}{m} \cdot v - g\cos\theta_{eq} \cdot \theta + \frac{1}{m}u \qquad (1)$$

$$\frac{d\theta}{dt} = \left(\frac{l}{m} + \frac{g\cos\theta_{eq}}{v_{eq}^2}\right) v + \frac{g\sin\theta_{eq}}{v_{eq}} \cdot \theta \qquad (2)$$

We will exploit that $0 = -mg\cos\theta_{eq} + lv_{eq}^2$, i.e., $\frac{g\cos\theta_{eq}}{v_{eq}^2} = \frac{l}{m}$, to write the normalised state-space linear equation in matrix form:

$$\frac{d}{dt}\begin{pmatrix} v \\ \theta \end{pmatrix} = \begin{pmatrix} -\frac{2d \cdot v_{eq}}{m} & -g\cos\theta_{eq} \\ +\frac{2l}{m} & +\frac{g\sin\theta_{eq}}{v_{eq}} \end{pmatrix} \begin{pmatrix} v \\ \theta \end{pmatrix} + \begin{pmatrix} \frac{1}{m} \\ 0 \end{pmatrix} u$$

This may be either stable or unstable, depending on l, d and v_{eq}.
*Si $l > 0$, $d > 0$, $\theta_{eq} \leq 0$ is stable.

Particular case: stability of level flight

$$\frac{d}{dt}\begin{pmatrix} v \\ \theta \end{pmatrix} = \begin{pmatrix} -\frac{2d\cdot v_{eq}}{m} & -g \\ +\frac{2l}{m} & 0 \end{pmatrix}\begin{pmatrix} v \\ \theta \end{pmatrix} + \begin{pmatrix} \frac{1}{m} \\ 0 \end{pmatrix} u$$

Characteristic equation:

$$s^2 + \frac{2dv_{eq}}{m}s + \frac{2lg}{m} = 0$$

substituting $l/m = g/v_{eq}^2$:

$$s^2 + \frac{2dv_{eq}}{m}s + \frac{2g^2}{v_{eq}^2} = 0$$

IF we write it as $s^2 + 2\zeta\omega_n s + \omega_n^2 = 0$, we get:

$\omega_n = \frac{\sqrt{2}g}{v_{eq}}$, y $2\zeta\sqrt{2}gv_{eq}^{-1} = 2dv_{eq}m^{-1}$, sale $\zeta = \frac{dv_{eq}^2}{mg\sqrt{2}} = \frac{dv_{eq}^2}{lv_{eq}^2\sqrt{2}} = \frac{d}{l\sqrt{2}}$

The natural freq. depends (inverse) on speed, damping depends on drag/lift ratio, which is usually small. Lightly damped oscillations.

Conclusions

- Longitudinal flight "if the moment of inertia is small" or/and "if the corrective torque against angular deviation is large" can approximate 2 DoF, order 2 model, in speeds. The plane as a "point".
- There is an equilibrium point $(\theta_{eq}, v_{eq}, u_{eq})$, which may be stable (glider or level flight) but if plane is efficient (d/l is small), it is very lightly damped, requiring pilot attention (or auto-pilot). High thrust produces ascending flight and, from a given value onwards, unstable phugoid dynamics.

Equilibrium point, linearization and local stability of an aircraft phugoid mode (simplified 2n order eqs.)

*This code ran without errors in Matlab **R2022b**

Video-presentation at: http://personales.upv.es/asala/YT/V/fugeqlinEN.html

Objectives: numerically, analyze the equilibrium points and their stability, of a simplified 2nd order model of the phugoid flight mode of an aircraft.

Table of Contents

Modeling and linearization

```
d=1/5; l=1; m=1; g=9.8;%invented numbers
```

Equations of phugoid movement

$$m\frac{dv}{dt} = -mg\sin\theta - dv^2 + u$$ [*tangential*]

$$mv\frac{d\theta}{dt} = -mg\cos\theta + l\cdot v^2$$ [*normal* (signed: positive is anticlocwise rotation)]

```
syms v theta U
StateEqs=[-g*sin(theta)-d*v^2/m+U/m;-g*cos(theta)/v+l*v/m];vpa(StateEqs,4)
```

ans =

$$\begin{pmatrix} -0.2\,v^2 + U - 9.8\sin(\theta) \\ v - \dfrac{9.8\cos(\theta)}{v} \end{pmatrix}$$

Equilibrium points

If we wish to compute the thrust and velocity for a certain desired glide angle θ_e, we would have:

$$0 = -mg\sin\theta_e - dv_e^2 + u$$

$$0 = -mg\cos\theta_e + l\cdot v_e^2$$

hence

$$v_e = \sqrt{\frac{mg}{l}\cos\theta_e}\ , \quad u_e = mg\cdot\left(\sin\theta_e + \frac{d}{l}\cos\theta_e\right)$$

Particular cases

```
umax_eq=sqrt(l^2+d^2)/l %maximum specific thrust for equilibrium to exist
```

```
umax_eq = 1.0198
```

Conditions for level flight $\theta_e = 0$:

$$v_e = \sqrt{\frac{mg}{l}}, \quad u_e = mg \cdot \frac{d}{l}$$

```
Thrust_level=d/l %specific thrust (Thrust/weight)
```

```
Thrust_level = 0.2000
```

Zero thrust (glider) equilibrium:

$$mg \sin\theta_e = -dv_e^2$$

$$mg \cos\theta_e = l \cdot v_e^2$$

o sea, $\tan\theta_e = -\frac{d}{l}$.

```
Theta_glider=atan(-d/l)*180/pi
```

```
Theta_glider = -11.3099
```

Of course, from either u_e or v_e we can solve for the other two unknowns (omitted for brevity).

Linearization (for stability analysis)

```
jacA=jacobian(StateEqs,[v,theta]);vpa(jacA,4) %we need to substitute
equilibrium point, of course
```

ans =

$$\begin{pmatrix} -0.4\,v & -9.8\cos(\theta) \\ \frac{9.8\cos(\theta)}{v^2} + 1.0 & \frac{9.8\sin(\theta)}{v} \end{pmatrix}$$

```
jacB=jacobian(StateEqs,U);vpa(jacB,4)
```

ans =

$$\begin{pmatrix} 1.0 \\ 0 \end{pmatrix}$$

Characteristic equations $det(sI - A) = 0$ comes from:

```
syms s
collect(simplify(det(s*eye(2)-jacA)), s) %Second order...
```

ans =

$$s^2 + \left(-\frac{245\,v\sin(\theta) - 10\,v^3}{25\,v^2}\right)s + \frac{245\,v^2\cos(\theta) - 98\sin(\theta)\,v^2 + 2401\cos(\theta)^2}{v^2\,25}$$

Wether it is stable or not, it depends... All the coefficients should be positive to have stability.

If operating point is "high climb angle" it may become unstable. We will verify such fact numerically.

Plots

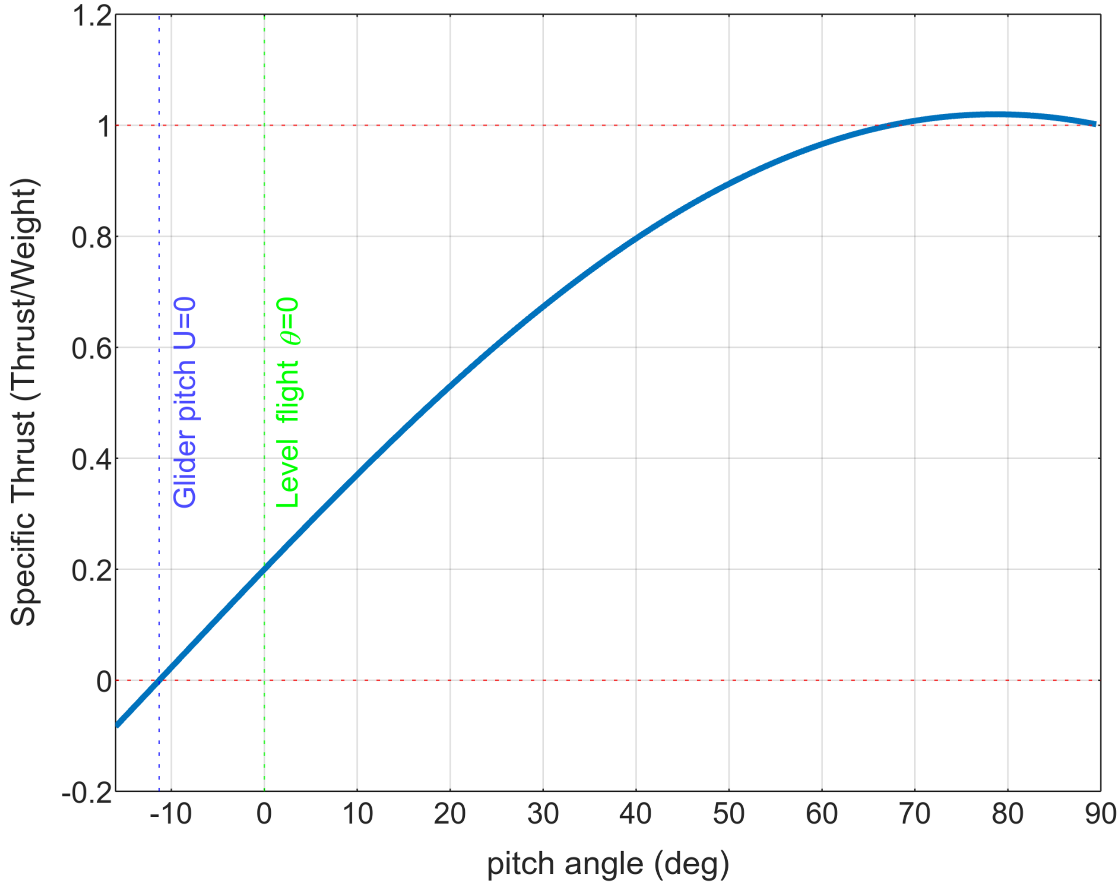
```
Thmin=round(Theta_glider-5);
Th_range=(Thmin:0.5:89.5)*pi/180; %Range of test angles
NE=length(Th_range);
```

Store results for plotting

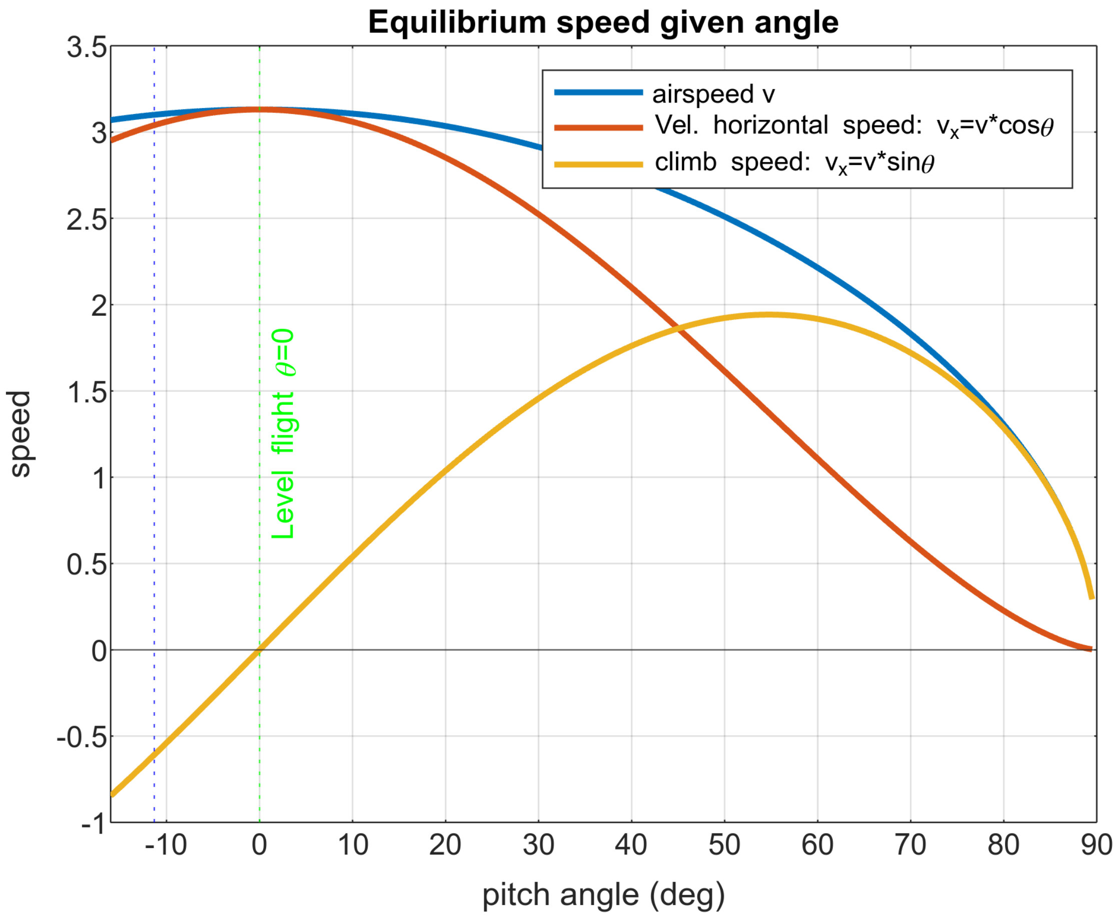
```
grv0=zeros(NE,1); grU=zeros(NE,1);
grwp=zeros(NE,1); grreal=zeros(NE,2);

for k=1:NE
    % operating point
    theta=Th_range(k);
    v0=sqrt(m*g*cos(theta)/l); %airspeed
    U0=m*g*(sin(theta)+d*cos(theta)/l); %thrust
    grv0(k)=v0;
    grU(k)=U0;
    % Stability analysis of \dot x=Ax
    A=eval(subs(jacA,{v},{v0})); %linearised model
    ee=eig(A); %eigenvalues
    grreal(k,:)=real(ee); %we store real part
    wp=imag(ee(1)); %and imaginary part (oscillation frequency)
    grwp(k)=wp;
end
```

```
plot(Th_range*180/pi,grU/m/g,LineWidth=2), grid on,
xlabel("pitch angle (deg)"), ylabel("Specific Thrust (Thrust/Weight)"),
xlim([Thmin 90]),
yline(1,':r'), yline(0,':r'),
xline(Theta_glider,':b',Label="Glider pitch U=0",
LabelVerticalAlignment="middle")
xline(0,':g',Label="Level flight \theta=0", LabelVerticalAlignment="middle")
```



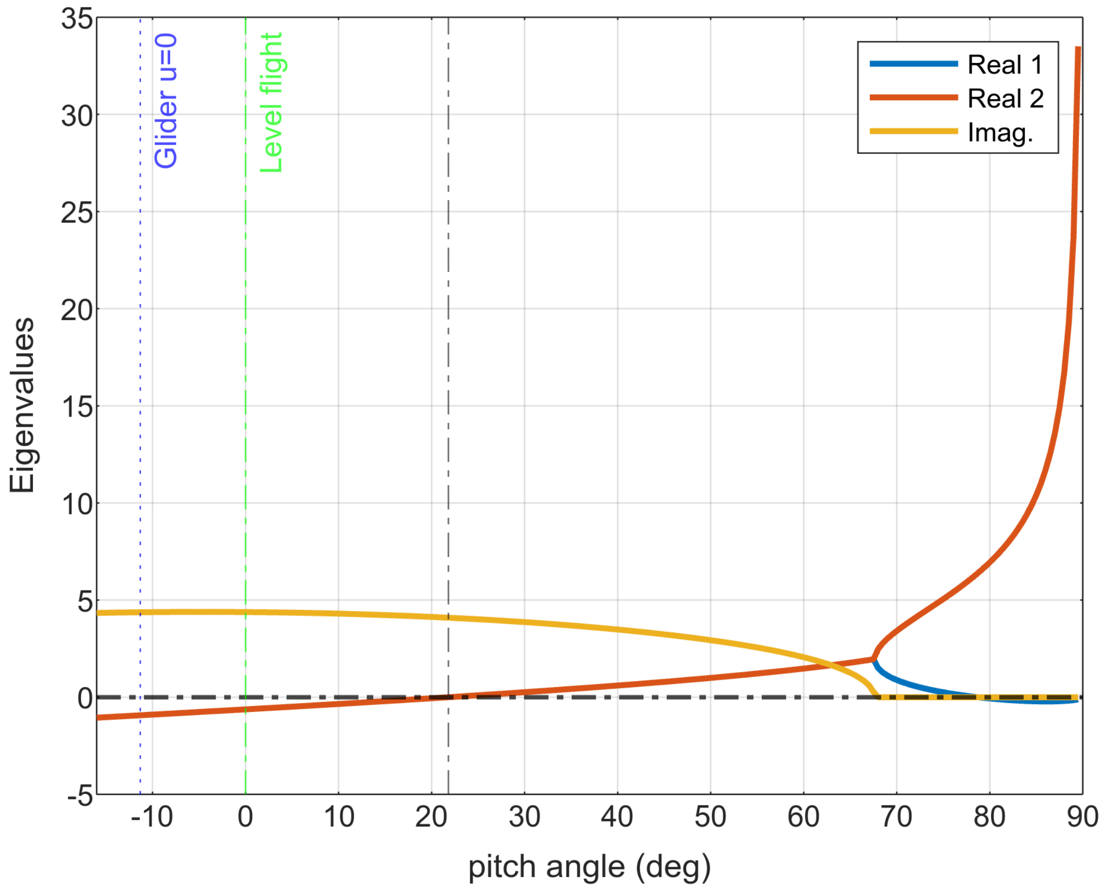

```
plot(Th_range*180/pi,grv0,LineWidth=2), grid on, xlabel("pitch angle (deg)"),
hold on, plot(Th_range*180/pi,grv0.*cos(Th_range'),LineWidth=2)
plot(Th_range*180/pi,grv0.*sin(Th_range'),LineWidth=2), hold off,
ylabel("speed"),xlim([Thmin 90]), xline(Theta_glider,':b'), yline(0)
xline(0,':g',Label="Level flight \theta=0", LabelVerticalAlignment="middle")
legend("airspeed v","Vel. horizontal speed: v_x=v*cos\theta","climb speed:
v_x=v*sin\theta")
title("Equilibrium speed given angle")
```




```
%Stability analysis plots (eigenvalues)
plot(Th_range*180/pi,[grreal grwp],LineWidth=2), grid on, xlabel("pitch
angle (deg)"), ylabel("Eigenvalues"),
xlim([Thmin 90]), xline(Theta_glider,':b',Label="Glider u=0"),
yline(0,"-.k",LineWidth=1.5)
angle_unstable=atan(2*d/l)*180/pi
```

```
angle_unstable = 21.8014
```

```
xline(0,'-.g',Label="Level flight")
xline(angle_unstable,'-.')
legend("Real 1","Real 2","Imag.")
```



• Control surfaces, flaps, etc. do change lift, drag, angle of incidence... We are dealing with simplified models for somehow close to level flight, subsonic regime. Aggressive maneuvers may require a 3GL model of the aircraft, including rotational moment of inertia, etc., nor have we taken into account that a propeller can vary its thrust with airspeed, etc. That is, everything in here is approximate, with target audience being "industrial" engineering students. Aerospace students will for sure deal with more complex (but more accurate) models in flight dynamics.