Matérn and squared-exponential kernels: intuition from a control/ signal processing engineer point of view

Video presentation at: https://personales.upv.es/asala/YT/V/maternEN.html

This code was run with Matlab **`R2024a`**

Objectives: understand the "filtering" intuition (spectral factor transfer function) on Matérn kernel and the squared-exponential one.

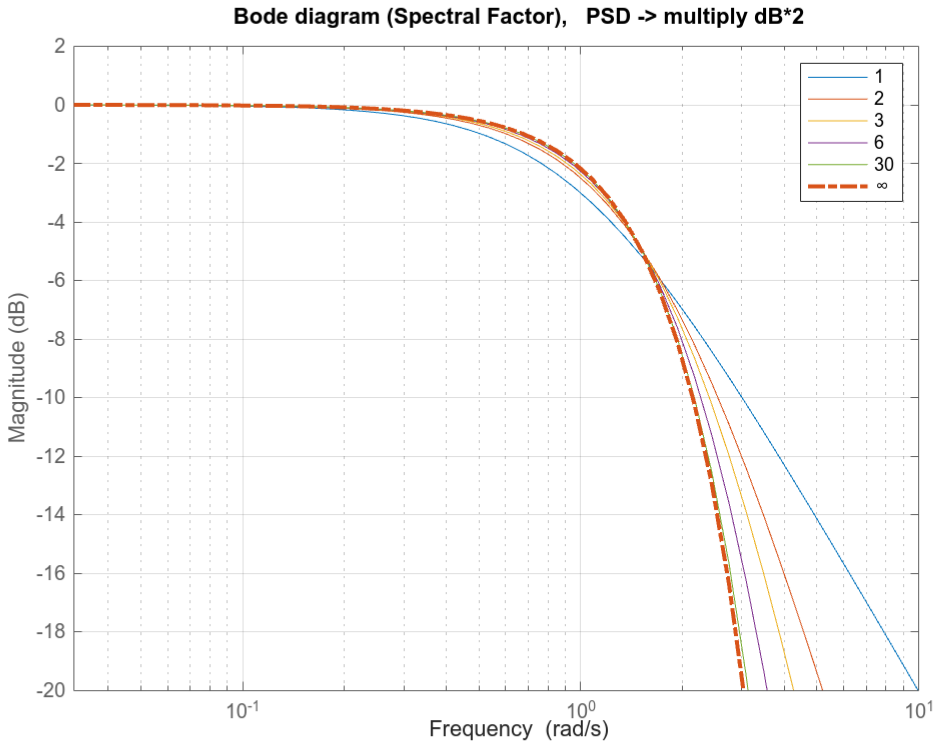
Frequency response

The "Matérn" kernel [from **Bertil Matérn** (1917 – 2007), Wikipedia] is a generalisation to maybe "fractional n" of white noise filtered by $\frac{A}{(\tau s+1)^n}$, well actually more similar to this $\frac{A}{(\frac{\rho}{\sqrt{2n-1}}s+1)^n}$... and in fact, A also changes with n so that the 2-norm (steady-state standard deviation) keeps constant to a pre-defined value σ.

But these issues involving Bessel functions, Gamma functions, etc. are not needed in detail in order to intuitively understand what we are doing if we come from a control systems background...

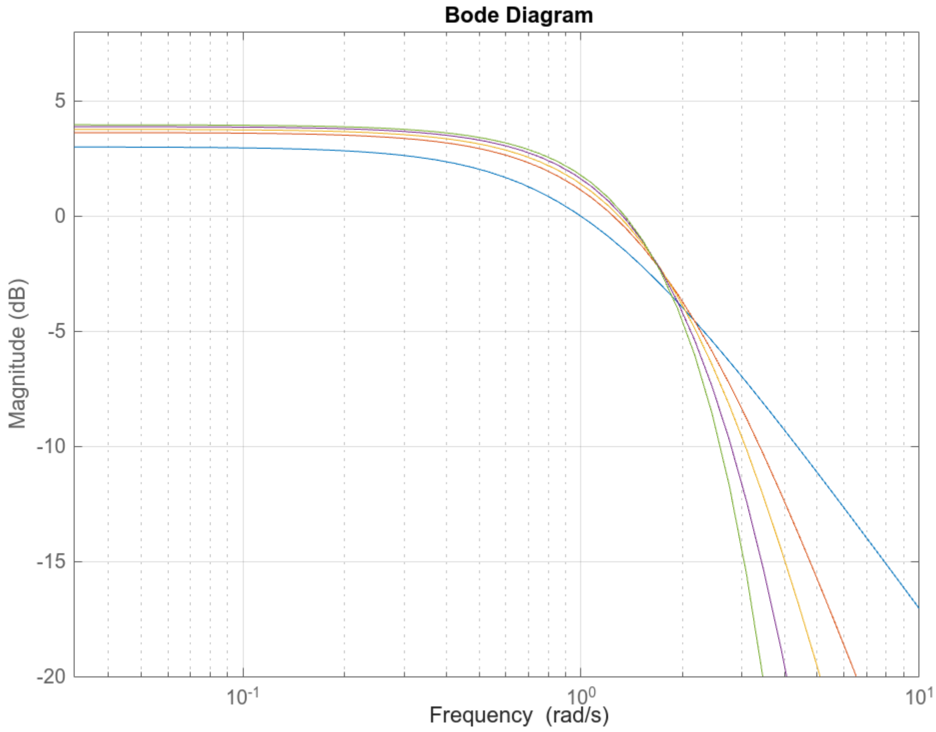
So, apart from "amplitude correction" to keep variance constant, we are doing something like this:

```
s=tf('s');
ntst=[1 2 3 6 30];
Gtst=cell(5,1);Gtst2=cell(5,1);
for k=1:5
    n=ntst(k);
    tau=1/sqrt(2*n-1);
    Gtst{k}=1/(tau*s+1)^n;
    Gtst2{k}=Gtst{k}/norm(Gtst{k});
end
w_tst=logspace(-1.5,1);
bodemag(Gtst{:},w_tst), grid on, ylim([-20 2])
title("Bode diagram (Spectral Factor),   PSD -> multiply dB*2")
hold on
plot(w_tst,20*log10(exp(-w_tst.^2/2/2)),"-.",LineWidth=2),
hold off
legend("1", "2", "3", "6", "30", "\infty")
```



This is the actual amplitude correction so the integral of the "power spectral density" (i.e., variance) keeps constant to one:

```
for k=1:length(Gtst)
    Gtst2{k}=Gtst{k}/norm(Gtst{k});
end
bodemag(Gtst2{:},logspace(-1.5,1)), grid on, ylim([-20 8])
```



Let us check Wikipedia formulae...

```
A=3;tau=1; n=5;
```

Causal spectral factor:

```
G=@(s) A/(tau*s+1)^n;
```

Power spectral density:

```
syms w real
psd=simplify(G(1j*w)'*G(1j*w),60)
```

psd =

$$\frac{9}{(w^2+1)^5}$$

Auto-covariance:

```
syms x real %distance between points
kk=simplify(ifourier(psd))
```

kk =

$$\frac{3\,e^{-|x|}\,(x^4+45\,x^2+10\,|x|^3+105\,|x|+105)}{256}$$

Integral of PSD is the variance, its. square root is standard deviation of the stationary process.

```
sg=simplify(sqrt(int(psd,-inf,inf))/sqrt(2*sym(pi))); %standard deviation
in steady state
```

```
eval(sg)
```

```
ans = 1.1093
```

```
sg1=norm(G(s)) %2-norm of a dynamic system
```

```
sg1 = 1.1093
```

```
eval(sqrt(subs(kk,0))) %autocovariance at zero distance is variancell=5+n;
```

```
ans = 1.1093
```

Now, Matérn kernel formulae from Wikipedia end up in the covariance:

```
nu=sym(n-1/2);
rho=tau*sqrt(2*nu);
kk2=simplify( sg^2*2^(1-nu)/gamma(nu)*(sqrt(2*nu)*x/
rho)^nu*besselk(nu,sqrt(2*nu)*x/rho) )
```

kk2 =

$$\frac{3\,e^{-x}\,(x^4+10\,x^3+45\,x^2+105\,x+105)}{256}$$

*Note that "`ifourier`" command above produced a polynomial with coefficients of odd degree with $|x|$, $|x|^3$, ... The expression `kk2` is supposed to be evaluated with $x \geq 0$ to yield identical results (it must be used with argument equal to the norm of the difference between two abscissa of the Gaussian process).

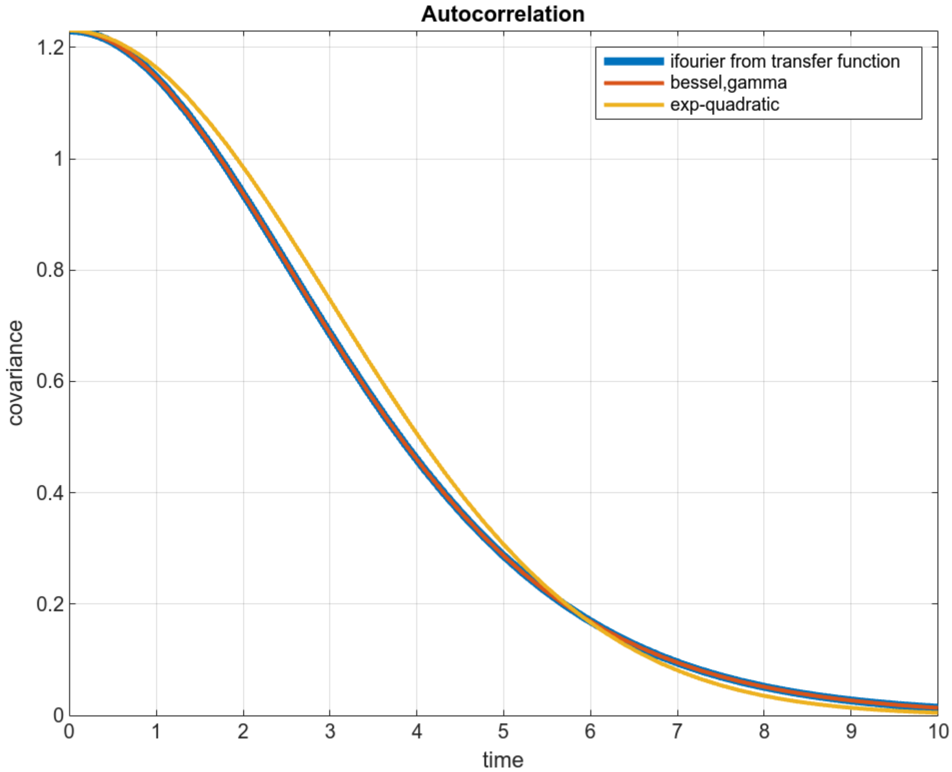
When $n \to \infty$ we get the "squared-exponential" kernel:

```
expq=exp(-(x/rho)^2/2)*sg^2
```

expq =

$$\frac{315\,e^{-\frac{x^2}{18}}}{256}$$

```
ll=5+n; %cosmetic, axis stuff
fplot(kk,[0, ll*tau],LineWidth=4), hold on, grid on
fplot([kk2 expq],[0, ll*tau], LineWidth=2), hold off
title("Autocorrelation")
legend("ifourier from transfer function","bessel,gamma","exp-quadratic")
xlabel("time"),ylabel("covariance")
```


Autocorrelation
ifourier from transfer function
bessel,gamma
exp-quadratic
covariance
time
0
0.2
0.4
0.6
0.8
1
1.2
0
1
2
3
4
5
6
7
8
9
10