Are complex numbers actually matrices?

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Video: https://personales.upv.es/asala/YT/V/imisMEN.html

Summary:

- **Complex Numbers as Matrices:** The field of complex numbers can be represented through 2x2 matrices. In this representation, the identity matrix acts as the real number **1**, and a specific matrix acts as the imaginary unit *i*.
- **Properties and Operations:** This matrix isomorphism preserves the properties of complex arithmetic. For instance, the **square of said matrix results in the negative identity matrix**, mimicking $i^2 = -1$. Additionally, the complex conjugate of a number is obtained by **transposing** its matrix representation, and the product of two such matrices follows the standard rules for multiplying complex numbers.
- **Euler's Formula and Derivatives: Euler's formula** (cos(*θ*)+*i*sin(*θ*)) and the properties of the exponential of an imaginary unit are accurately reflected in the matrix representation.
- **Infinitude of Imaginary Units:** there are **infinitely many matrices** that can function as imaginary units. We provide examples of various matrices—including 2x2 complex matrices (pure quaternions) and larger 4x4 matrices—that all satisfy the requirement of being an imaginary unit because their **square equals the negative identity matrix**.

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Geometry: 90º rotations

The "imaginary unit i" rotates 90º, anticlockwise, in the "complex plane".

The following matrix:

```
W=[0 -1;1 0]
```

```
W = 2×2
     0    -1
     1     0
```

rotates 90º in the Euclidean $\mathbb{R}^2$ plane.

Indeed:

```
syms x y real
Rotated=W*[x;y]
```

```
Rotated =
```

$$\begin{pmatrix} -y \\ x \end{pmatrix}$$

```
Rotated'*[x;y] %Orthogonal
```

```
ans = 0
```

Are they "the same"?

Algebra: addition, multiplication, division, polynomials...

Are complex numbers actually matrices? Are the two "90º rotations" two views of the same thing?

In a sense YES: the "field of complex numbers" can be represented as the linear span of eye(2) and the matrix

```
W=[0 -1;1 0]
```

```
W = 2×2
     0    -1
     1     0
```

[`Identity(2)` and `W`] act as [1 and "i"]. See that:

```
One=eye(2);
W^2 %minus "One"
```

```
ans = 2×2
    -1     0
     0    -1
```

Also, the inverse "$1/i$" is equal to "$-i$", because $(-i)\cdot i = -(-1) = 1$. The matrix inverse also does it:

```
inv(W)
```

```
ans = 2×2
     0     1
    -1     0
```

```
-W %Got it!
```

```
ans = 2×2
     0     1
    -1     0
```

A generic complex number $z = a + bi$ is represented as:

```
syms a b c d real
Z=(a*One+b*W)
```

```
Z =
```

$$\begin{pmatrix} a & -b \\ b & a \end{pmatrix}$$

```
%eig(Z) %the matrix encodes both the number and its conjugate, in a
certain way
```

The conjugate complex number $\bar{z} = a - bi$ is obtained by "transposing"

```
Z'
```

```
ans =
```

$$\begin{pmatrix} a & b \\ -b & a \end{pmatrix}$$

The product:

```
(a*One+b*W)*(c*One+d*W)
```

```
ans =
```

$$\begin{pmatrix} a\,c - b\,d & -a\,d - b\,c \\ a\,d + b\,c & a\,c - b\,d \end{pmatrix}$$

As expected:

```
(a*c-b*d)*One + (a*d+b*c)*W
```

```
ans =
```

$$\begin{pmatrix} a\,c - b\,d & -a\,d - b\,c \\ a\,d + b\,c & a\,c - b\,d \end{pmatrix}$$

More involved setups: exponentials and derivatives

Exponentials of complex numbers:

```
%Euler Formula
syms theta real
EE=exp(theta*1i)
```

```
EE =
```

$$e^{\theta\,i}$$

```
[real(EE),imag(EE)]
```

```
ans =
```

$$(\cos(\theta) \quad \sin(\theta))$$

What about building:

```
cos(theta)*One+sin(theta)*W
```

```
ans =
```

$$\begin{pmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{pmatrix}$$

We got it! It is the "matrix exponential":

```
R=expm(W*theta)
```

```
R =
```

$$\begin{pmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{pmatrix}$$

```
%expand(exp(a+b*1i))
%expm(a*One+b*W)
```

Derivatives

The derivative with respect to θ of $\rho(\theta) = e^{i\theta}$ is $ie^{i\theta} = i\rho(\theta)$.

```
dR=diff(R,theta)
```

```
dR =
```

$$\begin{pmatrix} -\sin(\theta) & -\cos(\theta) \\ \cos(\theta) & -\sin(\theta) \end{pmatrix}$$

```
W*R %here we have it!
```

```
ans =
```

$$\begin{pmatrix} -\sin(\theta) & -\cos(\theta) \\ \cos(\theta) & -\sin(\theta) \end{pmatrix}$$

Conclusion: "i" is not an "imaginary" number... it is a "very real" action (rotation geometry).

But "complex plane" and "real plane" are not the same

While the **Complex Plane** ($\mathbb{C}$) and the **Real Plane** ($\mathbb{R}^2$) look identical as sets of points (x, y) isomorphic to $x + yi$... but that isomorphism is only when considering sum and scaling by a real number; once additional operations are considered, they are very different "beings" algebraically.

The Real plane is a vector space (we can "add" and "scale" vectors), but the complex plane admits multiplication, division, roots... it is a "field", an algebraic structure with more operations on it than a plain "vector space".

Matrices are "transformations of the real plane", and not the "points of such a plane". So, a complex number is a "point" in $\mathbb{C}$ that acts like a "transformation" in $\mathbb{R}^2$.

Such transformations admit addition, multiplication and division (inverse matrix), but the points of $\mathbb{R}^2$ over which they act don't, they cannot be multiplied or divided when they are just considered an "Euclidean geometry plane". The points on which a "group/ring/field of transformations" acts are not the same as the "transformations themselves".

Nevertheless, these "matrices that act as complex numbers" are of course also a 2-dimensional vector space, a 2D subspace of the 4D space $\mathbb{R}^{2\times 2}$.

There is more ahead

This is just a "motivation" to better apprehend the fact that there are many matrices that act as "imaginary units". Infinitely many, in fact!

For instance:

```
Wnew=[1i 1+1i;-1+1i -1i]/sqrt(3)
```

```
Wnew = 2×2 complex
   0.0000 + 0.5774i   0.5774 + 0.5774i
  -0.5774 + 0.5774i   0.0000 - 0.5774i
```

```
Wnew^2
```

```
ans = 2×2
   -1.0000   -0.0000
   -0.0000   -1.0000
```

```
syms a b real
c=sqrt(1-a^2-b^2);
WW=[c*1i b+a*1i;-b+a*1i -c*1i] %This matrix receives the name of "pure quaternion"
```

```
WW =
```

$$\begin{pmatrix} \sqrt{-a^2-b^2+1}\,\mathrm{i} & b+a\,\mathrm{i} \\ -b+a\,\mathrm{i} & -\sqrt{-a^2-b^2+1}\,\mathrm{i} \end{pmatrix}$$

```
simplify(WW^2,100)
```

```
ans =
```

$$\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$$

```
eig(WW) %This is not just by chance...
```

```
ans =
```

$$\begin{pmatrix} -\mathrm{i} \\ \mathrm{i} \end{pmatrix}$$

These 2x2 complex matrices squaring to -1 are at the core of "quaternion" approach to rotations and the "spin" concept in quantum mechanics (Pauli matrices).

Or even "larger" matrices, such as these 4x4 ones, "Dirac" matrices.

```
W3=[zeros(2) W;W zeros(2)]
```

```
W3 = 4×4
     0     0     0    -1
     0     0     1     0
     0    -1     0     0
     1     0     0     0
```

```
W3^2
```

```
ans = 4×4
    -1     0     0     0
     0    -1     0     0
     0     0    -1     0
     0     0     0    -1
```

```
W4=[zeros(2) -diag([1 -1]);diag([1 -1]) zeros(2)]
```

```
W4 = 4×4
     0     0    -1     0
     0     0     0     1
     1     0     0     0
     0    -1     0     0
```

```
W4^2
```

```
ans = 4×4
    -1     0     0     0
     0    -1     0     0
     0     0    -1     0
     0     0     0    -1
```