Solving Linear Ordinary Differential Equations with Matlab's symbolic toolbox: spring-mass-damping example (**forced** response)

This code executed in Matlab **R2022a**

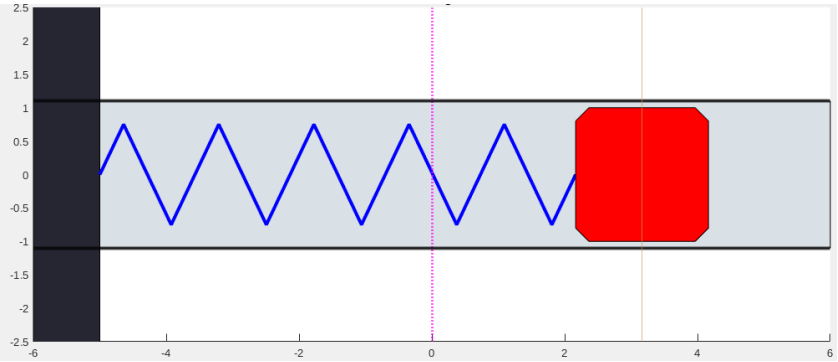
Video-presentation: http://personales.upv.es/asala/YT/V/masmusymForzEN.html

Objectives: analyzing the solution of linear ODEs provided by Matlab (symbolic toolbox **dsolve**), understanding its meaning, and graphically representing it in a spring-damper model, in this case with a forced response.

Table of Contents

EDO in non-normalized (second derivatives) form

Consider the following spring (with some damping, too, by friction with walls and spring losses):



Let us solve the equation of a mass-spring-damper system (linearized, zero equilibrium; linear viscous friction assumed), starting at initial conditions out of equilibrium, written as:

$$M\frac{d^2y}{dt^2} = -ky - b\frac{dy}{dt} + F(t)$$

```
syms y(t) F(t) %symbolic function of time
syms M k b real %constant parameters
vel=diff(y); % notation for velocity
accel=diff(y,2); % notation for acceleration

ODE_SYM =     M*accel == -k*y-b*vel+F(t)
```

ODE_SYM(t) =

$$M\,\frac{\partial^2}{\partial t^2}\,y(t) = -b\,\frac{\partial}{\partial t}\,y(t) + F(t) - k\,y(t)$$

```
%if it were a nonlinear ODE, a solution could not, in general, be found in the form of
M_num=0.5;k_num=2;b_num=0.2; %numerical values for constant parameters
%let's replace symbols by actual numbers
ODE_mass_spring=subs(ODE_SYM, {M,k,b}, {M_num,k_num,b_num})
```

ODE_mass_spring(t) =

$$\frac{\frac{\partial^2}{\partial t^2}\,y(t)}{2} = -\frac{\frac{\partial}{\partial t}\,y(t)}{5} + F(t) - 2\,y(t)$$

General solution

With no fixed initial conditions, there are infinitely many solutions, formally related to two integration constants and some integrals (convolutions) of force input:

```
sol=simplify(dsolve(ODE_mass_spring))
```

sol =

$$C_1\,\mathrm{e}^{-\frac{t}{5}}\,\sigma_2 - C_2\,\mathrm{e}^{-\frac{t}{5}}\,\sigma_1 - \frac{10\,\sqrt{11}\,\mathrm{e}^{-\frac{t}{5}}\,\sigma_2\int \mathrm{e}^{t/5}\,\sigma_1\,F(t)\,\mathrm{d}t}{33} + \frac{10\,\sqrt{11}\,\mathrm{e}^{-\frac{t}{5}}\,\sigma_1\int \mathrm{e}^{t/5}\,\sigma_2\,F(t)\,\mathrm{d}t}{33}$$

where

$$\sigma_1 = \sin\left(\frac{3\,\sqrt{11}\,t}{5}\right)$$

$$\sigma_2 = \cos\left(\frac{3\,\sqrt{11}\,t}{5}\right)$$

Particular solution for given initial conditions (position and speed) and input

```
ODE_mass_spring=subs(ODE_mass_spring,F(t),-2.5+6*sin(3*t));
position=simplify(dsolve(ODE_mass_spring, y(0)==3, vel(0)==0))
```

position =

$$\frac{12677\,\mathrm{e}^{-\frac{t}{5}}\cos\left(\frac{3\,\sqrt{11}\,t}{5}\right)}{2644} - \frac{1500\sin(3\,t)}{661} - \frac{360\cos(3\,t)}{661} + \frac{102677\,\sqrt{11}\,\mathrm{e}^{-\frac{t}{5}}\sin\left(\frac{3\,\sqrt{11}\,t}{5}\right)}{87252} - \frac{5}{4}$$

Let's check that the prescribed initial conditions are indeed met:

```
subs(position,t,0)
```

```
ans = 3
```

```
vel=simplify(diff(position))
```

```
vel =
```

$$\frac{1080\sin(3\,t)}{661}-\frac{4500\cos(3\,t)}{661}+\frac{4500\,\mathrm{e}^{-\frac{t}{5}}\cos\left(\frac{3\sqrt{11}\,t}{5}\right)}{661}-\frac{67885\sqrt{11}\,\mathrm{e}^{-\frac{t}{5}}\sin\left(\frac{3\sqrt{11}\,t}{5}\right)}{21813}$$

```
subs(vel,t,0)
```

```
ans = 0
```

Transients take more or less $e^{-\frac{t_{est}}{5}} \approx 0.02$, so:

```
t_est=log(0.02)*(-5)
```

```
t_est = 19.5601
```

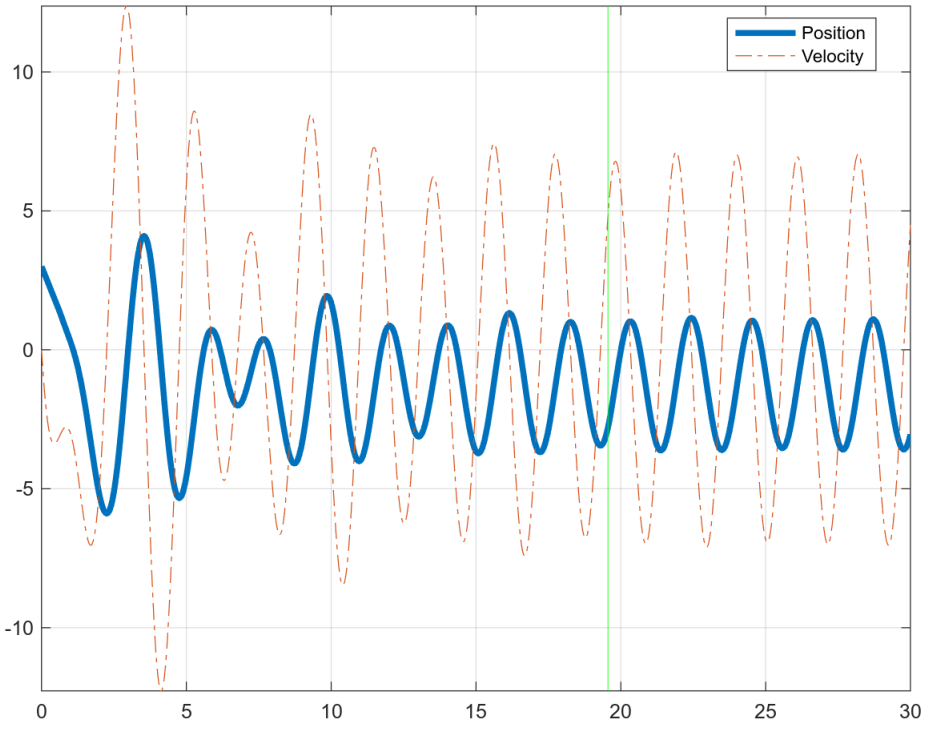
The system's proper oscillation frequency is:

```
omega_p=3*sqrt(11)/5
```

```
omega_p = 1.9900
```

If we draw position (thick blue) and speed (dashed red), we get:

```
Tf=30; %the final time at which we wish to stop plotting
fplot(position,[0 Tf],LineWidth=3), hold on
fplot(vel,[0 Tf],LineStyle='-.'), hold off, grid on,
xline(t_est,'g')
legend(["Position","Velocity"], location='best')
```


Position
Velocity
10
5
0
-5
-10
0
5
10
15
20
25
30