Planar rotations: complex plane representation

Antonio Sala

Lecture notes for robotics & aerospace engineering
Universitat Politècnica de València

UNIVERSITAT POLITÈCNICA DE VALÈNCIA

Outline

Motivation:

Exponential map of planar rotation results in an Euler formula with identity and a matrix that squares to minus identity.

Objectives:

Identify all relevant rotation Lie group concepts both in "matrix" world and in "complex number" world

Contents:

Review of exponential map. Table with equivalences between rotation matrix representation and complex number representation.

Exponential map

2D rotations were generated by a skew-symmetric matrix.
If rotational speed is constant (in a given frame), we get:
From $\dot{R} = \Omega R$ we get $R(t) = expm(\Omega t)R(0)$.
Constant angular speed maneuvers are seen in the "rotation matrix" language as matrix exponentials.

Every $R \in SO(2)$ is the exponential of a skew-symmetric matrix.

 ©2027 Antonio Sala

Proof of the exponential formula

$$\Omega = \theta J,\ J = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix};\quad J^2 = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} = -I,\quad J^3 = -J, \ldots$$

Euler formula: *cf. complex exponential: $e^{\theta j} = \cos\theta + \sin\theta \cdot j$

$$\begin{aligned} exp(\Omega) &= I + \Omega + \Omega^2/2! + \Omega^3/3! + \Omega^4/4! + \ldots \\ &= I + \theta J - \theta^2/2 \cdot I - \theta^3/3! \cdot J + \theta^4/4! \cdot I + \ldots \\ &= \left(1 - \theta^2/2! + \theta^4/4! + \ldots\right) * I + \left(\theta - \theta^3/3! + \theta^5/5! + \ldots\right) * J \\ &= \cos(\theta) \cdot I + \sin(\theta) \cdot J = \underbrace{\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}}_{R(\theta)} \end{aligned}$$

 ©2027 Antonio Sala

Group operations in "alternative" representations?

Rotation matrices have a "product" (i.e., the matrix product) that convert them to a Lie group (group+manifold), an "inverse" (transpose) and an "action" which is Rx (matrix vector product) which actually rotates "something x".

IF we don't "represent" rotations by matrices, but with "other things" we must provide a "product" and an "inverse" which is compatible with rotations: if we have some "mathematical stuff" ξ, associated to a rotation matrix $R(\xi)$, we should define a product $\xi_1\xi_2$ so that $R(\xi_1\xi_2) = R(\xi_1)R(\xi_2)$, and also $R(\xi^{-1}) = (R(\xi))^{-1}$, and an action $\xi(x) = R(\xi)x$.

 ©2027 Antonio Sala

The 2D rotation SO(2): angle representation

• On the 1-D S^1 representation, we can define $R(\theta)$ as the usual rotation matrix $[\cos\theta \ -\sin\theta; \sin\theta \ \cos\theta]$, and the "addition modulo 2π" is the nice "product" that preserves the rotation, and just changing sign of angle is the "inverse".
Action on a point of space needs a bit of trigonometry so it's sort of rebuilding the rotation matrix.

• As $e^{\theta J} = R(\theta)$, $R(\theta_1 + \theta_2) = R(\theta_1)R(\theta_2)$, and $R(-\theta) = R(\theta)^{-1}$, the angle representation is actually a **logarithmic** representation. If you confront with $\log(ab) = \log(a) + \log(b)$, $\log(1/a) = -\log(a)$, you see that θ has a "logarithmic" behaviour, informally $\theta \approx log(R)$.

 ©2027 Antonio Sala

SO(2): unit circle representation

- The $(\cos\theta, \sin\theta)$ corrdinates lie in something that with we have become accustomed to understand as the "**complex plane**", with the 2D basis being one symbol $\mathbf{1}$ and a second symbol $\mathbf{i}$ which squares to $-\mathbf{1}$.
Then complex number product (and division, i.e., inverse) is the sought operation that mimics rotation behaviour. Lie algebra are just "real numbers" and exponential map is $e^{j}\theta$.
The vector $(\cos\theta, \sin\theta)$ has modulus 1: rotations "are" unit circle imaginary numbers.

- The "action" over x is computed just considering x as a complex number and multiplying by a complex number in the unit circle.

 ©2027 Antonio Sala

SO(2): complex unit circle representation

	Matrices	Complex numbers
Rotation	$\mathrm{R}=\begin{pmatrix}\cos\theta & -\sin\theta\\ \sin\theta & \cos\theta\end{pmatrix}=\cos\theta I+\sin\theta J$	$z=\cos\theta+\sin\theta\mathbf{i}$ $q_0+q_1\mathbf{i}$, $q_0^2+q_1^2=1$
Identity	$\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}$	$1+0\mathbf{i}$
Inverse	$R^{-1}=R^T$	$z^{-1}=\bar{z}=\cos\theta-\sin\theta\mathbf{i}$
Group product	Matrix product	Complex product, $\mathbf{i}^2=-1$
Exponential map	$e^{\theta J}=e^{\theta\begin{pmatrix}0 & -1\\ 1 & 0\end{pmatrix}}$	$z=e^{\theta\mathbf{i}}$
Action on $x=[x_1;x_2]$	$R\cdot x$	$z\cdot(x_1+x_2\mathbf{i})$
Lie algebra $\mathfrak{so}(2)$	$\Omega=\omega\cdot J=\omega\cdot\begin{pmatrix}0 & -1\\ 1 & 0\end{pmatrix}$	$\omega\mathbf{i}$
Kinematics	$\dot{R}=\Omega R,\quad \dot{x}=\Omega x$	$\dot{z}=(\omega\mathbf{i})\cdot z,\quad \dot{x}=(\omega\mathbf{i})\cdot x$
Restoration of corrupt rotation	$[a\ b;c\ d]$ converts to $\cos\theta=(a+d)/2$, $\sin\theta=(c-b)/2$ plus scaling so $\cos^2+\sin^2=1$ *There is a more elegant SVD version.	$z_{true}=z/\lvert z\rvert$

©2027 Antonio Sala

UNIVERSITAT POLITÈCNICA DE VALÈNCIA

Conclusions

- 2D rotations are elegantly represented by the unit circle in the complex plane.
- The group of rotations can be equivalently represented with rotation matrices and matrix product or with unit-norm complex numbers and complex number product.
- Exponential of skew-symmetric matrices translates to exponential of a purely imaginary number.
- The action of said group (rotating a point in 2D space) is also represented as the product of a complex number times another one in the unit-circle.

*A generalization of complex numbers named "**quaternions**" will enable us to say the same for 3D rotations in future materials.

 ©2027 Antonio Sala