

# 2D Rotations: Geometry, Trig Identities, Matrices

Introductory material for aerospace/robotics engineers

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## Videos:

<http://personales.upv.es/asala/YT/V/rot2d1EN.html> (definitions),

<http://personales.upv.es/asala/YT/V/rot2d2EN.html> (properties),

<http://personales.upv.es/asala/YT/V/rot2d3EN.html> (trigonometry back from properties),

<http://personales.upv.es/asala/YT/V/rot2d4EN.html> (kinematics),

<http://personales.upv.es/asala/YT/V/rot2d5EN.html> (exponential map).



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# Outline

## Motivation:

Understanding rotations is very important in robotics and aerospace (end-effector orientation, attitude control maneuvers,...). Before tackling the 9 elements of a 3D rotation matrix, we must master the 2D case.

## Objectives:

We derive the rotation matrix from fundamental trigonometric sum identities and analyze its structure.

## Contents:

Polar coordinates, rotation of coordinates, the  $SO(2)$  matrix, and small-angle approximations.



# Motivation: Why 2D Rotations first?

- ▶ In aerospace/robotics, many problems are effectively planar or can be decomposed into planar rotations:
  - **Navigation:** Heading angle (yaw) over a flat-earth local map.
  - **Aerodynamics:** Angle of attack ( $\alpha$ ) is a planar definition.
  - **Flight Control:** Longitudinal vs. Lateral dynamics separation.
  - **Robotics:** each joint in a robot arm carries out a planar rotation.
- ▶ 3D rotations are eventually built as a **sequence** of these 2D rotations (Euler angles). Also, exponential formulae for 2D generalise (with some tweaks) to 3D (Rodrigues).
- ▶ Understanding the properties of a  $2 \times 2$  rotation block is a convenient preliminary step to fully understanding  $3 \times 3$  rotation matrices.



## Geometry: Mapping a point in 2D

Mapping a point from a frame  $x_b$  (body frame) to  $x_w$  (world), both sharing the same origin:

$$\begin{pmatrix} x_w \\ y_w \end{pmatrix} = \text{Rot} \left( \begin{pmatrix} x_b \\ y_b \end{pmatrix} \right)$$

Using polar coordinates, a point at distance  $r$  and initial angle  $\phi$ :

- **Initial state:**  $x_b = r \cos \phi$ ,  $y_b = r \sin \phi$
- **Rotated state (by angle  $\theta$ ):**

$$x_w = r \cos(\phi + \theta) \quad y_w = r \sin(\phi + \theta)$$

► The goal is to express  $(x_w, y_w)$  as a linear function of  $(x_b, y_b)$  with rotation angle  $\theta$  as parameter.



# The Fundamental Trigonometric Expansion

Applying the sum-of-angles identities:

$$\textcircled{1} \quad x_w = r \cos(\phi + \theta) = r(\cos \phi \cos \theta - \sin \phi \sin \theta)$$

$$\textcircled{2} \quad y_w = r \sin(\phi + \theta) = r(\sin \phi \cos \theta + \cos \phi \sin \theta)$$

Substituting the original coordinates  $x_b = r \cos \phi$  and  $y_b = r \sin \phi$ :

$$\bullet \quad x_w = (x_b) \cos \theta - (y_b) \sin \theta$$

$$\bullet \quad y_w = (x_b) \sin \theta + (y_b) \cos \theta$$

⇒ This is a linear transformation! Can be expressed in matrix form.



## The 2D Rotation Matrix $R(\theta)$

We can now write the previous equations in matrix-vector form:

$$\begin{pmatrix} x_w \\ y_w \end{pmatrix} = \underbrace{\begin{bmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{bmatrix}}_{R(\theta)} \begin{pmatrix} x_b \\ y_b \end{pmatrix}$$

**Structure of  $R(\theta)$ :**

- **Column 1:**  $\begin{pmatrix} \cos \theta \\ \sin \theta \end{pmatrix}$  is the **world** position of the body forward  $X$ -axis unit vector  $(1,0)$ .
- **Column 2:**  $\begin{pmatrix} -\sin \theta \\ \cos \theta \end{pmatrix}$  is the **world** position of the body left  $Y$ -axis unit vector  $(0,1)$ .

\*Note: a positive  $\theta$  corresponds to a Counter-Clockwise (CCW) rotation.

► This group of transformations of the plane is called  $SO(2)$ . It has **1 degree of freedom** ( $\theta$ ), “just 1 way to turn”.



# Properties of $R \in \mathbb{R}^{2 \times 2}$ , $R \in SO(2)$

- **Orthogonality:**  $R^T R = I$ , also  $RR^T = I$  [transpose is inverse]. Check:

$$\begin{bmatrix} c & s \\ -s & c \end{bmatrix} \begin{bmatrix} c & -s \\ s & c \end{bmatrix} = \begin{bmatrix} c^2 + s^2 & 0 \\ 0 & s^2 + c^2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

- **Determinant:**  $\det(R) = \cos^2 \theta - (-\sin^2 \theta) = 1$ .
- **Eigenvalues:**  $\text{eig}(R) = \{e^{\pm i\theta}\}$ . Eigenvectors are complex-valued.
- **Commutativity (Unique to 2D!):** From angle sum trig identities, we can prove that in 2D, the order does not matter, and that combining rotations is equivalent to “adding the angles”:

$$R(\theta_1)R(\theta_2) = R(\theta_2)R(\theta_1) = R(\theta_1 + \theta_2)$$

\*Note that “multiplication in matrix world” corresponds to “addition in angle world”: **angle units are sort of a “logarithm” of rotations** (cf.  $10^3 \cdot 10^2 = 10^{3+2}$ ). This will allow defining an “**exponential map**” from angles to rotation matrices.

# Abstracting the concept of rotation away from trigonometry

(will ease going to 3 and higher dimensions)

- Orthogonality can be interpreted as “**metric and angle preservation**”:  $\|x_w\|^2 = \|Rx_b\|^2 = x_b^T R^T R x_b = x_b^T x_b = \|x_b\|^2$ . If metric is identity we easily get key properties of “higher-dimensional rotations”. Other metrics change to  $R^T G R = G$ .
- Determinant equal to 1 amounts to “**orientation preservation**”. The rotated “y” vector must be “counterclockwise” to the rotated “x” one, if they were so before rotation. This also will scale up to 3D as preserving “right-handed orthogonal frames”.
- **Commutativity** must be **dropped** for 3 or higher dimensions. This will be good... we may “yaw” a plane with just “pitch” and “roll” actions, richer geometry.



# Trigonometry as a consequence of norm preservation

Consider any matrix

$$R = \begin{pmatrix} a & q \\ b & d \end{pmatrix}, \quad I = R^T R = \begin{pmatrix} a & b \\ q & d \end{pmatrix} \begin{pmatrix} a & q \\ b & d \end{pmatrix}$$

this entails  $a^2 + b^2 = 1$ ,  $q^2 + d^2 = 1$ ,  $aq + bd = 0$ .

- 3 equations, 4 unknowns: 1 degree of freedom only.
- Determinant = 1, i.e.,  $ad - bq = 1$ , 4 equations and 4 unknowns, zero DoF? **NO**.

That equation is implied by the fact that, for any four numbers

$$(a^2 + b^2)(q^2 + d^2) = (aq + bd)^2 + (ad - bq)^2$$

Thus, we get  $1 \cdot 1 = 0 + (ad - bq)^2$ , so either  $ad - bq = 1$  (rotation) or  $ad - bq = -1$  (reflection), but then it is not a “new” equation but a discrete choice between two solutions of the original three equations.



$$R = \begin{pmatrix} a & q \\ b & d \end{pmatrix}, \quad R^T R = I$$

Columns must be norm 1, and column 1 must be orthogonal to column 2. In 2D, the only way  $(q, d)$  is orthogonal to  $(a, b)$  is if  $(q, d) = \lambda \cdot (-b, a)$ . But they have the same norm, so  $\lambda = 1$  or  $\lambda = -1$ . It is  $\lambda = 1$  the one making determinant 1, so

$$R = \begin{pmatrix} a & -b \\ b & a \end{pmatrix}$$

From  $a^2 + b^2 = 1$ ; there exist some  $\theta$  so  $a = \cos \theta$  and  $b = \sin \theta$ . We chose cosine for  $a$  so that rotating “zero” degrees gives rise to identity and positive theta is “clockwise”. This is it, we derived:

$$R(\theta) = \begin{pmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{pmatrix}$$

from just  $R^T R = I$ ,  $\det(R) = +1$  and reference frame  $R(\theta = 0) = I$ .



## The "Small Angle" Approximation

Small increments of angle around a given rotation  $\theta_0$  result in:

$$dR = \begin{pmatrix} -\sin(\theta_0) & -\cos(\theta_0) \\ \cos(\theta_0) & -\sin(\theta_0) \end{pmatrix} \cdot d\theta$$

In aerospace stability and control, we often assume  $\theta$  small:

- $\cos \theta \approx 1$
- $\sin \theta \approx \theta$

The linearized rotation matrix around  $R = I$  becomes:

$$R(\theta) \approx I + \left. \frac{dR}{d\theta} \right|_{R=I} d\theta = \begin{bmatrix} 1 & -\theta \\ \theta & 1 \end{bmatrix} = I + \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \cdot \theta$$

This is the basis of **linearized attitude dynamics**. If  $\theta$  is small, the transformation is almost the identity plus a skew-symmetric term related to the angular displacement. The skew-symmetric matrix is not by chance, see next slide...

# Time-varying trajectories $R(t)$ : rotational motion

Consider a trajectory with  $\dot{\theta} = \omega$ , constant angular speed.

- From  $RR^T = I$  we have  $0 = \frac{d}{dt}(RR^T) = \frac{dR}{dt}R^T + R\frac{dR^T}{dt}$ . Thus, denoting  $\Omega = \frac{dR}{dt}R^T$ , we have  $\Omega + \Omega^T = 0$  so it must be a skew-symmetric matrix  $\Omega$ , denoted as the “angular speed matrix”.
- In the 2D case,

$$\frac{dR}{dt}R^T = \omega \cdot \begin{pmatrix} -s & -c \\ c & -s \end{pmatrix} \begin{pmatrix} c & s \\ -s & c \end{pmatrix} = \begin{pmatrix} 0 & -\omega \\ \omega & 0 \end{pmatrix}$$

In a more abstract language, matrix  $J = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$  is called the (infinitesimal) **generator** of motions, or the **basis** of the Lie algebra (logarithm of rotation matrices, see later).



## Velocity field

- From  $\Omega = \frac{dR}{dt} R^T = \frac{dR}{dt} R^{-1}$  we get  $\frac{dR}{dt} = \Omega R$ .
- If we consider  $x_w(t) = R(t)x_b$ , with constant “body position” of “body things” (I don’t move with respect to myself: my car isn’t moving, it’s the world who is moving right when I turn left) then

$$\frac{dx_w}{dt} = \frac{dR}{dt} x_b = \Omega R x_b = \Omega x_w$$

so  $\Omega$  generates the “velocity field”. It is a matrix version of  $v = \omega \times r$  in vector products in 3D, to be discussed in depth in other materials. For instance, the point  $(2, 2)$  rotating at  $\omega = 5$  rad/s around the origin results in having a velocity of  $(-10; 10)$ , i.e., a vector of modulus  $\omega r$ ,  $r = \sqrt{2^2 + 2^2}$  perpendicular to  $(2, 2)$ .

- In a more abstract language, matrix  $J = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$  is called the (infinitesimal) **generator** of motions, or the **basis** of the Lie algebra (logarithm of rotation matrices, see later).



## Velocity is orthogonal to position: rotates $90^\circ$ and scales

As  $\Omega$  is skew symmetric,  $\Omega + \Omega^T = 0$  we have that:

$$x_w^T \cdot \frac{dx_w}{dt} = x_w^T \Omega x_w = 0$$

because  $x_w^T \Omega x_w + (x_w^T \Omega x_w)^T = x_w^T (\Omega + \Omega^T) x_w = 0$ .

Note also that  $\Omega^T \Omega = \omega^2 I$ , so our “unit rotation speed matrix” with  $\omega = 1$  **is a rotation itself** that converts  $x_w$  to something orthogonal to it.

Indeed, it is a  $90^\circ$  rotation if  $\omega = 1$ :

$$\Omega = \begin{pmatrix} \cos(90^\circ) & -\sin(90^\circ) \\ \sin(90^\circ) & \cos(90^\circ) \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} = J$$



## Acceleration field

If we take one further derivative to  $\frac{dx_w}{dt} = \Omega x_w$ , we get:

$$\frac{d^2 x_w}{dt^2} = \frac{d\Omega}{dt} x_w + \Omega \frac{dx_w}{dt} = \left( \frac{d\Omega}{dt} + \Omega^2 \right) x_w$$

If we call the angular acceleration  $\alpha := \frac{d\Omega}{dt}$ , then:

$$\frac{d^2 x_w}{dt^2} = \left( \frac{d\Omega}{dt} + \Omega^2 \right) x_w = \left( \alpha \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} + \omega^2 \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} \right) \cdot x_w = (\alpha J - \omega^2 I) x_w$$

- The **skew-symmetric** term  $\alpha J x_w$  is the **tangential** acceleration, orthogonal to  $x_w$ , parallel to velocity  $\Omega x_w$ .
- The **symmetric** term  $-\omega^2 x_w$  is the **normal** acceleration, centripetal, in the same direction as  $x_w$  but pointing towards the center of rotation.

# Exponential map

- From  $\Omega = \frac{dR}{dt} R^T = \frac{dR}{dt} R^{-1}$  we get  $\frac{dR}{dt} = \Omega R$ , hence  $R(t) = e^{\Omega t} \cdot R(0)$ , i.e., a *matrix exponential* formula.
- Any rotation can be obtained from initial frame  $R(0) = I$  as the exponential of a matrix, which is called the **exponential map** of  $SO(2)$ . Indeed, if I rotate at  $\theta$  radians per second, in 1 second I will reach angle  $\theta$ ; abusing the notation changing argument from “time” to “angle”, we write:

$$R(\theta) = \text{expm} \left( \begin{pmatrix} 0 & -\theta \\ \theta & 0 \end{pmatrix} \right) = e^{J\theta}$$

Again, there is just one degree of freedom, one “way to smoothly change to one rotation to another nearby one”: angular rate  $\omega$  because the *Lie algebra*  $\dot{R}R^{-1} = \text{span}(J)$  is unidimensional.



# “Time” isn’t actually needed

- Motion “in time” and “velocities” and “accelerations” are interesting and didactic for a robotics engineering background.
- But we can easily check that

$$\frac{dR}{d\theta} = J \cdot R$$

hence, if  $R(0) = I$ , we have the exponential solution  $R(\theta) = e^{J\theta}$  without involving “time”, independent on the path we have taken to reach angle  $\theta$  and its speed.



# Proof of the exponential formula

$$\Omega = \theta J, \quad J = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}; \quad J^2 = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} = -I, \quad J^3 = -J, \dots$$

**Euler formula:**

\*cf. complex exponential:  $e^{\theta j} = \cos \theta + \sin \theta \cdot j$

$$\exp(\Omega) = I + \Omega + \Omega^2/2! + \Omega^3/3! + \Omega^4/4! + \dots$$

$$= I + \theta J - \theta^2/2 \cdot I - \theta^3/3! \cdot J + \theta^4/4! \cdot I + \dots$$

$$= (1 - \theta^2/2! + \theta^4/4! + \dots) \cdot I + (\theta - \theta^3/3! + \theta^5/5! + \dots) \cdot J$$

$$= \cos(\theta) \cdot I + \sin(\theta) \cdot J = \underbrace{\begin{pmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{pmatrix}}_{R(\theta)}$$

**Euler-Rodrigues formula:** as  $J^2 = -I$ ,  $\cos \theta \cdot I = I + (1 - \cos \theta)J^2$ , so  $\exp(\theta J) = I + \sin \theta J + (1 - \cos \theta)J^2$ . Basically useless in 2D, very useful in 3D.



## Summary and Transition to 3D

- A 2D rotation is a **linear operator** (matrix) that preserves length and orientation. Matrix columns represent the **rotated unit basis vectors**.
- It is represented by a **single parameter**  $\theta$ . It has an **exponential map**  $R(\theta) = \exp(J\theta)$ , generated by  $J = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$ .
- We can derive metric and orientation preservation from trigonometric identities or, vice-versa, derive the trigonometric formula for its elements from the metric and orientation preservation constraints.
- **Looking ahead:** To rotate in 3D in the XY plane, letting the Z coordinate invariant (rotation "axis"), we simply "embed" this 2D block into a  $3 \times 3$  matrix:

$$R_z(\theta) = \begin{bmatrix} \cos \theta & -\sin \theta & 0 \\ \sin \theta & \cos \theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

