

2D Rotations: Geometry, Trig Identities, Matrices

Introductory material for aerospace/robotics engineers

Antonio Sala,
Universitat Politècnica de València

Vídeos: <http://personales.upv.es/asala/YT/V/rot2d1EN.html>, <http://personales.upv.es/asala/YT/V/rot2d2EN.html>



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Outline

Motivation:

Understanding rotations is very important in robotics and aerospace (end-effector orientation, attitude control maneuvers,...). Before tackling the 9 elements of a 3D rotation matrix, we must master the 2D case.

Objectives:

We derive the rotation matrix from fundamental trigonometric sum identities and analyze its structure.

Contents:

Polar coordinates, rotation of coordinates, the $SO(2)$ matrix, and small-angle approximations.



Motivation: Why 2D Rotations first?

- ▶ In aerospace/robotics, many problems are effectively planar or can be decomposed into planar rotations:
 - **Navigation:** Heading angle (yaw) over a flat-earth local map.
 - **Aerodynamics:** Angle of attack (α) is a planar definition.
 - **Flight Control:** Longitudinal vs. Lateral dynamics separation.
 - **Robotics:** each joint in a robot arm carries out a planar rotation.
- ▶ 3D rotations are eventually built as a **sequence** of these 2D rotations (Euler angles). Also, exponential formulae for 2D generalise (with some tweaks) to 3D (Rodrigues).
- ▶ Understanding the properties of a 2×2 rotation block is a convenient preliminary step to fully understanding 3×3 rotation matrices.



Geometry: Mapping a point in 2D

Mapping a point from a frame x_b (body frame) to x_w (world), both sharing the same origin:

$$\begin{pmatrix} x_w \\ y_w \end{pmatrix} = \text{Rot} \left(\begin{pmatrix} x_b \\ y_b \end{pmatrix} \right)$$

Using polar coordinates, a point at distance r and initial angle ϕ :

- **Initial state:** $x_b = r \cos \phi$, $y_b = r \sin \phi$
- **Rotated state (by angle θ):**

$$x_w = r \cos(\phi + \theta) \quad y_w = r \sin(\phi + \theta)$$

► The goal is to express (x_w, y_w) as a linear function of (x_b, y_b) with rotation angle θ as parameter.



The Fundamental Trigonometric Expansion

Applying the sum-of-angles identities:

$$\textcircled{1} \quad x_w = r \cos(\phi + \theta) = r(\cos \phi \cos \theta - \sin \phi \sin \theta)$$

$$\textcircled{2} \quad y_w = r \sin(\phi + \theta) = r(\sin \phi \cos \theta + \cos \phi \sin \theta)$$

Substituting the original coordinates $x_b = r \cos \phi$ and $y_b = r \sin \phi$:

$$\bullet \quad x_w = (x_b) \cos \theta - (y_b) \sin \theta$$

$$\bullet \quad y_w = (x_b) \sin \theta + (y_b) \cos \theta$$

⇒ This is a linear transformation! Can be expressed in matrix form.



The 2D Rotation Matrix $R(\theta)$

We can now write the previous equations in matrix-vector form:

$$\begin{pmatrix} x_w \\ y_w \end{pmatrix} = \underbrace{\begin{bmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{bmatrix}}_{R(\theta)} \begin{pmatrix} x_b \\ y_b \end{pmatrix}$$

Structure of $R(\theta)$:

- **Column 1:** $\begin{pmatrix} \cos \theta \\ \sin \theta \end{pmatrix}$ is the **world** position of the body forward X -axis unit vector $(1,0)$.
- **Column 2:** $\begin{pmatrix} -\sin \theta \\ \cos \theta \end{pmatrix}$ is the **world** position of the body left Y -axis unit vector $(0,1)$.

*Note: a positive θ corresponds to a Counter-Clockwise (CCW) rotation.

► This group of transformations of the plane is called $SO(2)$. It has **1 degree of freedom** (θ), “just 1 way to turn”.



Properties of $R \in \mathbb{R}^{2 \times 2}$, $R \in SO(2)$

- **Orthogonality:** $R^T R = I$, also $RR^T = I$ [transpose is inverse]. Check:

$$\begin{bmatrix} c & s \\ -s & c \end{bmatrix} \begin{bmatrix} c & -s \\ s & c \end{bmatrix} = \begin{bmatrix} c^2 + s^2 & 0 \\ 0 & s^2 + c^2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

- **Determinant:** $\det(R) = \cos^2 \theta - (-\sin^2 \theta) = 1$.
- **Eigenvalues:** $\text{eig}(R) = \{e^{\pm i\theta}\}$. Eigenvectors are complex-valued.
- **Commutativity (Unique to 2D!):** From angle sum trig identities, we can prove that in 2D, the order does not matter, and that combining rotations is equivalent to “adding the angles”:

$$R(\theta_1)R(\theta_2) = R(\theta_2)R(\theta_1) = R(\theta_1 + \theta_2)$$

*Note that “multiplication in matrix world” corresponds to “addition in angle world”: **angle units are sort of a “logarithm” of rotations** (cf. $10^3 \cdot 10^2 = 10^{3+2}$). This will allow defining an “**exponential map**” from angles to rotation matrices.

Abstracting the concept of rotation away from trigonometry

(will ease going to 3 and higher dimensions)

- Orthogonality can be interpreted as “**metric and angle preservation**”: $\|x_w\|^2 = \|Rx_b\|^2 = x_b^T R^T R x_b = x_b^T x_b = \|x_b\|^2$. If metric is identity we easily get key properties of “higher-dimensional rotations”. Other metrics change to $R^T G R = G$.
- Determinant equal to 1 amounts to “**orientation preservation**”. The rotated “y” vector must be “counterclockwise” to the rotated “x” one, if they were so before rotation. This also will scale up to 3D as preserving “right-handed orthogonal frames”.
- **Commutativity** must be **dropped** for 3 or higher dimensions. This will be good... we may “yaw” a plane with just “pitch” and “roll” actions, richer geometry.

