Conditional independence: definition, examples

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Video-presentations:

http://personales.upv.es/asala/YT/V/condin1EN.html , http://personales.upv.es/asala/YT/V/condin2EN.html

Objectives: defining conditional independence and understanding its meaning in intuitive sense, as well as its importance in modelling for physics and control.

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Conditional Independence: formal definition

Suppose we have three random variables a, b, c.

- b and c are said to be "conditionally independent when $a = a_m$ " if

$$p(b|c = c_m, a = a_m) = p(b|a = a_m)$$

or

$$p(c|b = b_m, a = a_m) = p(c|a = a_m).$$

Equivalently, if $p(b, c|a = a_m) = p(b|a = a_m) \cdot p(c|a = a_m)$.

- If b and c and are conditionally independent given any possible observation of a, then we say that they are "conditionally independent given a".

In "plain" informal terms, they are cond. independent given a "**if b and c seem independent when the value a of is known**".

Examples

Thanks: Wikipedia and "towardsdatascience.com".

- "b: Power of a car" and "c: average annual cost of speeding tickets" are "statistically dependent", but knowing "a: speed when ticketed" makes them "conditionally independent";

- **[independent does not imply cond. independent]** Two rolls of dice (variables "b" and "c") are "statistically independent", but they are no longer independent if we know the variable "a" which tells us that "their sum was 11"; then the relationship is "deterministic" (known one, know the other). They are independent, but not "conditionally independent given their sum".
- **[dependent does not imply cond. dependent]** Height (random variable "b") and "having a college degree" (random variable "c") are "statistically dependent" variables... "tall" people have more college degrees than "short" people (because children are among the shortest population); They are conditionally independent if we condition that they have a certain "age" or age interval (age is the variable "a").

- **[Sensors with measurement noise]** A "measurement m" of some "physical" variable x with "independent measurement noise" is assumed to be conditionally independent of "any other signal in the system" given x.

- **[State space model]** A discrete dynamical system

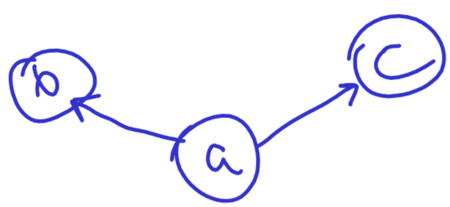
$$x_{k+1} = Ax_k + w_k, \qquad y_k = Cx_k + v_k,$$

being w_k the "process noise" and being v_k the "measurement noise", such that v_k and W_k are statistically independent (and also independent of previous values of themselves, white noise), verifies that x_{k+1} and y_k are " conditionally independent given x_k".

- **[Markov Hypothesis]** "Future" and "past" are conditionally independent given "present".

"Conditional independence" is the idea underlying "Bayesian Networks" used in artificial intelligence.

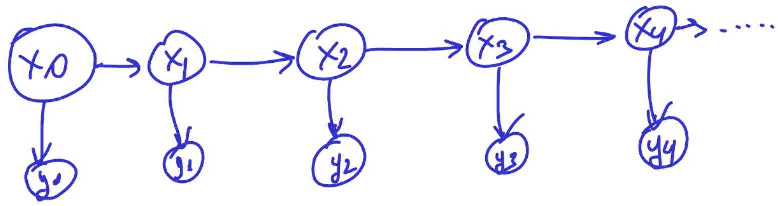
This relationship between the variables allows them to be expressed in graphic models like:



where the absence of "arrow" between b and c indicates "conditionally independent given a"... well, I say it more precisely below.

*In many cases of Bayesian Networks, the process is "reverse": certain variables are "known" or "assumed" to be conditionally independent to generate the statistical model of conditional probabilities, covariances, etc. saving time to think on unneeded probabilities and allowing for efficient computations.

The "Bayesian network" associated with the discrete-time state space model above, starting at $k = 0$, is:



Those networks usually draw a sort of "tree" where each variable is conditionally independent of "everything other than its children" given its "parents". ["cycles" in a generic non-tree directed graph do complicate the computations and ease of understanding of the meaning of such a graph].

Conclusions

The concept of "conditional independence" means that <two "dependent" variables happen to be "independent" if a third is known>. Finding this type of relationship is "key" in science so as not to infer "spurious correlation/causality" relationships.

- "Yellow teeth" and "lung cancer" have a strong correlation... which disappears if you condition to either smoker or non-smoker condition. Careful check of conditional independence avoids recommending "whitening toothpaste" to avoid lung cancer.

- "COVID-19 infection lethality" and "age" are highly correlated. The one who discovers that they are conditionally independent once the "degradation of the activity of the protein TT43-K associated with Xxxx's disease and the KT51b mutation of chromosome 11" is taken into account will win the Nobel Prize in medicine. Nothing is very clear as of today (March 2022), so all elderly people are vaccinated three or four times until something else more precise is discovered.

- Conditional independence is behind "Bayesian Networks" in Artificial Intelligence. It allows estimating "intermediate variables" and "belief propagation" in those networks.
- Conditional independence is behind the concept of "state" in Physics and control theory, and Bayesian filters/estimators in control and digital signal processing.