Derivative/Gradient info of a Stochastic Process: stationary case

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Video presentation: http://personales.upv.es/asala/YT/V/gradgpstEN.html

Objective: we may have observations of the gradient of a stochastic process (measurements of, say, position and speed) so we can improve position measurements, or we might wish to estimate such gradient from position measurements.

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Preliminaries

Let us consider a stochastic process where $f(x)$ is a random variable, for $x \in \mathbb{R}^n$, and consider that we have a "mean function" $\bar{f}(x)$, plus a cross-covariance (kernel) function $k(x, x') : \mathbb{R}^n \times \mathbb{R}^n \mapsto \mathbb{R}$.

This makes possible for us to write the conditional mean and variance of the process given observations of f at a given set of points (Kriging, best linear prediction, Kernel regression; these ideas are developed in other materials in this collection).

Under some "technical" conditions $f(x)$ may be differentiable...

Mean (expected value) of the derivatives

In the sequel e_i will denote the canonical vector in the i-th direction, $i = 1, \ldots, n$, i.e., the one with all components equal to zero except its i-th component, being that one equal to 1.

Gradient "mean" is just the gradient of the mean function:

$$E\left[\frac{\partial f}{\partial x_i}(x)\right] = E[\lim_{h\to 0}\frac{1}{h}(f(x+he_i) - f(x))] = \lim_{h\to 0}\frac{1}{h}(E[f(x+he_i)] - E[f(x)]) = \frac{\partial}{\partial x_i}E[f(x)]$$

Variance and covariance of the estimated derivatives

Let us consider a stochastic process with covariance kernel $k(x, x^{'})$.

- Let us consider covariance between the stochastic process $f(x)$ and its partial derivatives $\frac{\partial f}{\partial x_i}(x^{'})$ at an arbitrary different point $x^{'}$.

$$cov(f(x), \frac{\partial f}{\partial x_i}(x^{'})) = \frac{\partial k(x, x^{'})}{\partial x_i^{'}}$$

$$cov(\frac{\partial f}{\partial x_i}(x), f(x^{'})) = \frac{\partial k(x, x^{'})}{\partial x_i}$$

- Let us now consider covariance between partial derivatives

$$cov\left[\frac{\partial f}{\partial x_i}(x), \frac{\partial f}{\partial x_j}(x^{'})\right] = \frac{\partial^2 k}{\partial x_i \partial x_j^{'}}(x, x^{'})$$

Particularisation to stationary processes: statistical properties of its (partial) derivatives

- regarding covariance of a stationary process and its first derivatives, if $k(x, x^{'}) = \kappa(x^{'} - x)$, being $\kappa(h)$ a given autocovariance function (necessary, it must be an even function, i.e., $k(x, x^{'}) = \kappa(h) \equiv \kappa(-h) = k(x^{'}, x)$), depending only on the "difference" between points $h = x^{'} - x$.

Then

$$cov(f(x), \frac{\partial f}{\partial x_i}(x^{'})) = cov(f(x), \frac{\partial f}{\partial x_i}(x + h)) = \frac{\partial k(x, x^{'})}{\partial x_i^{'}} = \frac{\partial k(x, x + h)}{\partial h_i} = \frac{\partial \kappa}{\partial h_i}(h)$$

$$cov(\frac{\partial f}{\partial x_i}(x), f(x^{'})) = cov(\frac{\partial f}{\partial x_i}(x), f(x + h)) = cov(\frac{\partial f}{\partial x_i}(x^{'} - h), f(x^{'})) = \frac{\partial k(x, x^{'})}{\partial x_i} = \frac{\partial k(x^{'} - h, x^{'})}{\partial h_i} = -\frac{\partial \kappa}{\partial h_i}(h)$$

*Note that there is no problem with signs, because if $\kappa(h) = \kappa(-h)$ then $\frac{\partial \kappa}{\partial h_k}(h) = -\frac{\partial \kappa}{\partial h_k}(-h)$. Hence, the second formula can be continued saying that

$$cov(\frac{\partial f}{\partial x_k}(x), f(x')) = -\frac{\partial \kappa}{\partial h_k}(h) = \frac{\partial \kappa}{\partial h_k}(-h) = cov(f(x'), \frac{\partial f}{\partial x_k}(x' - h))$$

where the first formula has been applied at the last equality, so both formulae are basically the same for even $\kappa(\cdot)$ and, of course, covariances are symmetric.

In summary

$$cov(f(x), \frac{\partial f}{\partial x_i}(x+h)) = -\frac{\partial \kappa}{\partial h_i}(-h) = \frac{\partial \kappa}{\partial h_i}(h)$$

$$cov(f(x+h), \frac{\partial f}{\partial x_i}(x)) = -\frac{\partial \kappa}{\partial h_i}(h)$$

- Regarding covariance between partial derivatives, we have

$$cov\left[\frac{\partial f}{\partial x_i}(x), \frac{\partial f}{\partial x_j}(x')\right] = cov\left[\frac{\partial f}{\partial x_i}(x), \frac{\partial f}{\partial x_j}(x+h)\right] = \frac{\partial^2 k}{\partial x_i \partial x'_j}(x, x') = \frac{\partial^2 \kappa(x' - x)}{\partial x_i \partial x'_j} = -\frac{\partial^2 \kappa}{\partial h_i \partial h_j}(h)$$

In simpler terms:

$$cov\left[\frac{\partial f}{\partial x_i}(x), \frac{\partial f}{\partial x_j}(x+h)\right] = -\frac{\partial^2 \kappa}{\partial h_i \partial h_j}(h)$$

Stationary SISO time series (partial derivatives not needed then)

If $x \in \mathbb{R}$ and we change it to t (usual choice for "time"), partial derivatives get converted to plain ordinary derivatives, and h is understood as a time difference. We would then have:

$$cov(f(t_2), \frac{df}{dt}(t_1)) = -\frac{d\kappa}{dh}(t_2 - t_1) = \frac{d\kappa}{dh}(t_1 - t_2)$$

$$cov\left(\frac{df}{dt}(t_1), \frac{df}{dt}(t_2)\right) = -\frac{d^2\kappa}{dh^2}(t_1 - t_2)$$

In frequency domain (stationary)

We'll consider SISO case, for simplicity, otherwise we would have a "vector" of spatial frequencies.

Filtering by an ideal differentiator with transfer function "s" amounts to multiplying the power spectral density by $j\omega \cdot (-j\omega) = \omega^2$

So, for instance, the autocovariance of the derivative of a real GP will be:
`ifourier(w^2*fourier(kappa))`

and the covariance with the function that enters the differentiator would be
`ifourier(jw*fourier(kappa))`

*Of course, some "existence" conditions should hold: the covariance $\kappa(h) = e^{-|h|}$ has an spectral factor $G(s) = \sqrt{2}/(s+1)$, *i.e.*, it's generated by white noise filtered by $G(s)$.

```
G=@(s) sqrt(2)/(s+1);
syms w
simplify(ifourier(simplify(G(-1j*w)*G(1j*w))))
```

ans = $\mathrm{e}^{-|x|}$

its derivative has not finite variance:

```
simplify(ifourier(simplify(G(-1j*w)*G(1j*w)*w^2)))
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ans = $2\,\delta(x) - \mathrm{e}^{-|x|}$

We might have guessed that from the fact that $\kappa(h)$ is not differentiable at zero, so not twice differentiable... unless, yes, "impulsional" elements are allowed in derivatives.