Stochastic processes: “engineering” definition and examples

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Outline

Motivation:

Time signals with "noise" are of great importance in control and signal processing. The value of the output at each "instant" is a random variable.

Objectives:

Understand the concept of stochastic process that formalizes the idea of "signal with noise" and the ways to approach its study.

Contents:

Definition. Examples. How to study them. Conclusions.

The concept of an stochastic process

A “stochastic process” $\mathcal{X}$ is a set of random variables that take values in the same sample space Ω; there exists a random. vble for each element of an “index” set or domain, T.

It can be thought of as a “random function” from T to Ω.

What T is the “time”, is the formalization of the signals, eg ,outputs of a “processes to be controlled”, that contain noise.

Notation:

$$\mathcal{X} := \{X(t), t \in T\}, \quad X(t) \in \Omega$$

Examples:

- Noisy discrete-time signal $T \equiv \{0, 1, 2, \dots\}$(number of each sample), $\Omega = \mathbb{R}$, random function $T \mapsto \mathbb{R}$.
- Noisy continuous-time signal $T \equiv \mathbb{R}$ (time), $\Omega = \mathbb{R}$
- Multivariate random signal $T \equiv \mathbb{R}$(time), $\Omega = \mathbb{R}^m$... or $R \times m \mapsto R$.
- Noise in a 2000×1000 pixels photograph: $T \equiv \{1, \dots, 2000\} \times \{1, \dots, 1000\}$ (bidimensional spatial index variable), $\Omega = \mathbb{R}$ (b/w) or $\mathbb{R}^3$ (RGB).
- Noise in a 20fps RGB video recording: $T \equiv \{0, 1/20, 2/20, \dots\} \times \{1, \dots, 2000\} \times \{1, \dots, 1000\}$ (time $\times$ bidimensional spatial index variable), $\Omega = \mathbb{R}^3$.
- Vibrations of a beam excited by random forces: $T \equiv \mathbb{R} \times [0, L]$, $\Omega = \mathbb{R}$.

Objectives of the analysis of stochastic processes

In between of "**total randomness**" (white noise, $X(t_1)$ statistically indep. of $X(t_2)$ if $t_1 \neq t_2$), and "**determinism**" ($X(t) = sin(t)$) there are many intermediate situations.

The "abstract" objective of the analysis is to understand the relationships (dependence, covariance, correlation) between the variables at different values of t (time instants, image pixels, ...).

Applications:

- Signal filtering/smoothing, noise removal in process control, audio, video.
- Prediction: given the values at certain "past" instants, give a prediction of the values at "future" instants
- Interpolation: predict values at "intermediate" points (t with "spatial" meaning; [left-right, up-down] instead of future-past).
- Statistics-based control: if there exist (deterministic) manipulated "inputs" that change means, variances or correlations, use them to fulfill some goals.

How to study these processes

1. "descriptive" framework:

- Define mean $\mu(t) := E[X(t)]$, variance $\Sigma_t := E((X(t) - \mu(t))^2)$, covariance $R(t_1, t_2) := E((X(t_1) - \mu(t_1))(X(t_2) - \mu(t_2)))$
- Compute them from samples in "identification", "tuning" or "learning" phases.
- Use them to predict some variables as function of others when in "production". For instance, if T is time, predict "future" values from "past" ones.

How to study these processes

2. Physics-based framework:

- Assume that there is a certain dynamical system subject to random inputs that is the “cause” of the correlation at different instants of time, and determine the properties of the “integral” of that dynamics. ´

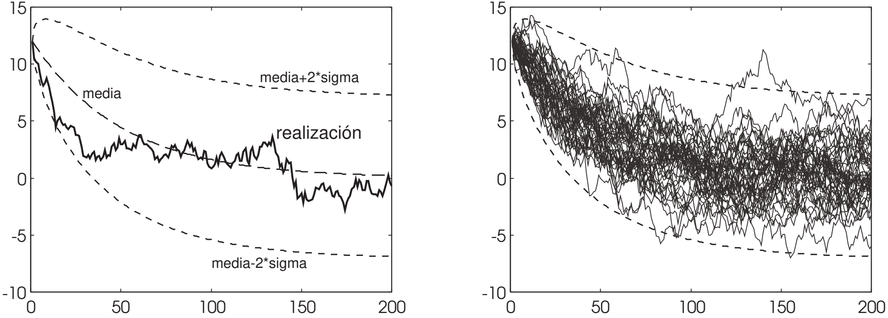
$$x_{k+1} = Ax_k + Bw_k, \qquad dx = Adt + BdW$$

  Car suspension in random bumpy road, flutter of a wing due to wind, random electromagnetic noise captured by an antenna.

- Other physics-based analysis write PDE equations for the “diffusion of the probability mass” [Fokker-Planck equations].

Example

Capacitor discharge from initial 12V (there is leakage resistance):



Mean, variance (or standard dev.) depend on time. They can be computed by “repetition of the experiment” (sample mean, etc.), *data-based* approach, or by *integration* of the differential equations of an RC circuit (in its *stochastic* version).

Conclusions

- The concepts of process and measurement noise, in time signals for control, audio, video, or in multidimensional signals (images) are formalized as stochastic processes (random functions): set of random variables, random function over an index set that means time (control) space (photo), or both (video, multivariable or PDE control).
- The practical usefulness is based on computing means, variances, correlations between different temporal instants/spatial positions to make predictions, filtering, denoising, etc.
- Computations can be made based on data [identification, learning, interpolation] or based on models [stochastic differential equations (cont.) or differences (discr.), probability-diffusion PDE, . . .].