Introduction to Projection Matrices

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Video-presentación disponible en:

personales.upv.es/asala/videos/**maprEN**.html

Outline

Motivation:

Projection in 2D or 3D have a clear geometric interpretation. We can extend such interpretation to arbitrary dimensions, to use in modelling, statistics (least squares fit, subspace ID, . . .).

Objectives:

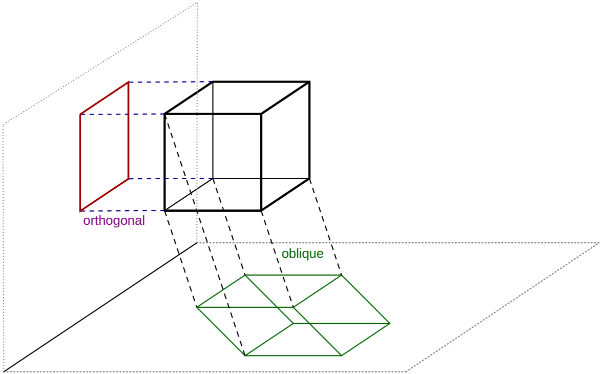
Understand projection matrices and their basic properties.

Contents:

2D and 3D projection. Projection matrix definition. Properties. Examples (particularly, pseudo-inverse). Conclusions.

2D/3D graphical interpretation of projection

Projections: orthogonal and oblique



Picture is derivative work of https://commons.wikimedia.org/wiki/File:Projection_oblique.svg

How to generalise to higher-dimensionality setups?

Preliminaries

A matrix $A_{n\times m}$ transforms a vector $x \in \mathbb{R}^m$ in another one $y = Ax \in \mathbb{R}^n$.

The column space of $A_{n\times m}$ is the subspace generated by the linear combination of its columns $col(A) := \{y : \exists\xi \text{ such that } y = A\xi\}$.

Scalar (inner) product of two vectors $< x, y >:= x^T y$.

Two (nonzero) vectors are orthogonal iff $x^T y = 0$.

The Euclidean norm of a vector (2-norm) is $\|x\| := \sqrt{x^T x}$.

Basic Idea: A projection P over a subspace $\mathcal{S}$ leaves unchanged the vectors that are already on $\mathcal{S}$, i.e., $P\xi = \xi\ \forall\xi \in \mathcal{S}$, and transforms vectors x not lying in $\mathcal{S}$ to vectors $\xi = Px \in \mathcal{S}$. Hence, $P\underbrace{(Px)}_{\xi} = \underbrace{Px}_{\xi}$.

Note: We will study projections onto vector subspaces ($0 \in \mathcal{S}$) not onto "affine" constructs (lines, hyperplanes, etc. not containing the origin).

Projections

A matrix $P_{n\times n}$ is said to be a **projection** one if $P^2 = P$. $[P(Px) = Px]$
The complement $P^c := I - P$ is also a projection matrix
$(I-P)(I-P) = I - 2P + P^2 = I - P$.

Trivially, every $x \in \mathbb{R}^n$ can be expressed as its projection Px plus its complement $P^c x$; indeed $x = Px + P^c x$.

A projection is **orthogonal** if Px and $P^c x$ are orthogonal for all x, i.e., $x^T P^T P^c x = 0$, which entails $P^T P_c = P^T(I-P) = 0$.

A non-orthogonal projection is said to be an **oblique** projection.

Projections: properties

Eigenvalues and eigenvectors: Any projection (be it orthogonal or oblique), if $Pv = \lambda v$ then $P^2 v = P\lambda v = \lambda^2 v$. As $Pv = P^2 v$, $\lambda v = \lambda^2 v$. With $v \neq 0$, it can only occur with $\lambda = 0$ or $\lambda = 1$.

Symmetry: A projection matrix is orthogonal if and only if it is symmetric:
[Orth⇒ Sym] $0 = P^T(I - P) = P^T - P^T P = P - P^T P$, con lo que $P^T P = P = P^T$.
[Sym⇒ Orth] $P^T(I - P) = P(I - P) = P - P^2 = P - P = 0$.

If P is an orthogonal projection, $\|x\|^2 = \|Px\|^2 + \|P^c x\|^2$.
Indeed,

$$(Px + P^c x)^T (Px + P^c x) = x^T P^T P x + x^T \underbrace{(P^T P^c + (P^c)^T P)}_{0} x + x^T (P^c)^T P^c x$$

Example 1

Consider:

$$P = \begin{pmatrix} 1.0000 & 0 & 0 \\ 0.4000 & -0.6000 & -1.2000 \\ -0.2000 & 0.8000 & 1.6000 \end{pmatrix}$$

It's a projection:

$$P^2 = \begin{pmatrix} 1.0000 & 0 & 0 \\ 0.4000 & -0.6000 & -1.2000 \\ -0.2000 & 0.8000 & 1.6000 \end{pmatrix} = P$$

But it is not an orthogonal one, i.e., it's oblique; indeed, P is not symmetric and

$$P^T \cdot (I_{3\times 3} - P) = \begin{pmatrix} -0.2000 & 0.8000 & 0.6000 \\ 0.4000 & -1.6000 & -1.2000 \\ 0.8000 & -3.2000 & -2.4000 \end{pmatrix} \neq 0$$

Projection of $x = (1,2,3)^T$ is $\xi = Px = (1,-4.4,6.2)^T$. Projection of ξ is $P\xi = P^2x = \xi$. But the complement $x - \xi$ and ξ are not orthogonal vectors.

Example 2

The **pseudoinverse** matrix of $A_{n\times m}$, $m \le n$, $rank(A) = m$, is
$A^{\dagger}_{m\times n} := (A^T A)^{-1} A^T$.

Matrix $P_{n\times n} := A_{n\times m} A^{\dagger}_{m\times n} = A(A^T A)^{-1} A^T$ has rank m, and it is a **projection**.

Indeed $P^2 = A(A^T A)^{-1} A^T A(A^T A)^{-1} A^T = A(A^T A)^{-1} A^T = P$.

It is an **orthogonal** projection, because P is a symmetric matrix. It projects (orthogonally) over the column space of A: if $x = A\mu$,
$Px = A(A^T A)^{-1} A^T A\mu = A\mu = x$.

Matlab:

```
A=[1 2;-2 2;3 -2];
P=A*pinv(A)
P^2 %same as P
P'*(eye(3)-P) %zero
```

Exercise (proposed): Check that, given $A_{n\times m}$ and $B_{n\times(n-m)}$, such that $[A\ B]$ is invertible, then matrix

$$P := [A\ B] \cdot \begin{pmatrix} I_{m\times m} & 0 \\ 0 & 0 \end{pmatrix} \cdot [A\ B]^{-1}$$

is a projection. P is denoted as the projection (oblique if $B^T A \neq 0$) over column space of A in the direction of columns of B.

Conclusions

- Projection matrices generalise the "intuition" in 2D and 3D to any dimension.
- P is a projection iff $P^2 = P$.
- Orthogonal projection, iff $P^2 = P$ and $P^T(I - P) = 0$; it implies symmetry of P.
- Eigenvalues in $\{0, 1\}$.
  - The subspace over which we are projecting is the one given by linear combinations of eigenvectors associated to unity eigenvalues, which render unaltered.