ν-gap metric: generalisation from real to complex/MIMO case (no proofs)

Antonio Sala

Control of complex systems

DISA – Universitat Politècnica de València

Video-presentación disponible en:

http://personales.upv.es/asala/YT/V/**gapm2EN**.html

Introduction

Motivation: Given a controller, `ncfmargin(P,K)` returns the robustness margin against *normalised coprime factor uncertainty* $P = (N + \Delta_n)(D + \Delta_d)^{-1}$. Given two plants, $P_1(j\omega)$ and $P_2(j\omega)$, the concept might be difficult to understand and compute.

Objectives: Understand how to translate ncf robustness to bounds in the uncertain plant's frequency response. Understand how to assert if P_2 will be stabilised by a given controller that achieves a given margin with P_1 (sufficient cond.). Understand the geometric interpretation of Vinnicombe's ν-gap.

(no proofs, informal)

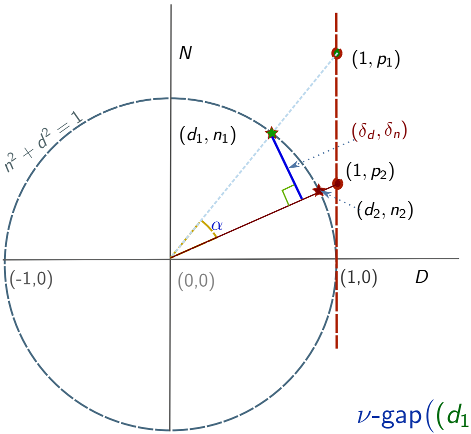
Vinnicombe, G.; *Uncertainty and Feedback, $\mathcal{H}_\infty$ Loop-shaping and the ν-gap Metric*, Imperial College Press (2000)

Real-valued geometrical interpretation (1)

We will represent $p = n/d$, $d \neq 0$, as $(d, n) \in \mathbb{R}^2$.

Equivalent fractions $(d_a, n_a) \equiv (d_b, n_b)$ iff $n_a/d_a = n_b/d_b$.

Normalized factorisation iff $n^2 + d^2 = 1$.



Any $p_1 \in \mathbb{R}$, interpreted as $p_1/1$, can be "projected" (normalized) to the circumference with center 0 radius 1, denoted as $\mathcal{B}$.

$$p_1/1 \text{ verifies } (1, p_1) \equiv \Big(\overbrace{\frac{1}{\sqrt{1+p_1^2}}}^{d_1}, \overbrace{\frac{p_1}{\sqrt{1+p_1^2}}}^{n_1}\Big)$$

Given $(d_1, n_1) \in \mathcal{B}$, the smallest (δ_d, δ_n) such that $(d_1 + \delta_d, n_1 + \delta_n) \equiv (d_2, n_2) \in \mathcal{B}$ is the difference between (d_1, n_1) and its orth. projection onto line $(\gamma d_2, \gamma n_2)$, $\gamma \in \mathbb{R}$. We will denote:

$$\nu\text{-gap}((d_1, n_1), (d_2, n_2)) := \|(\delta_d, \delta_n)\| = |\sin(\alpha)|$$

Normalised/unnormalised expressions of ν-gap

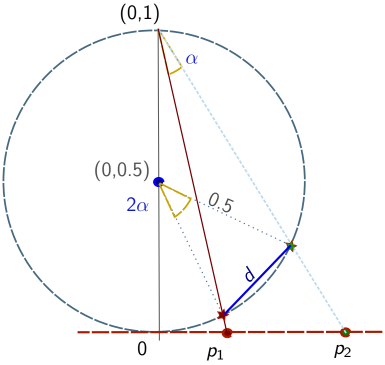
Scalar product of two norm-1 vectors is the *cosine* of the angle between them. Considering the perpendicular to (d_2, n_2), i.e., $(-n_2, d_2)$, then, assuming normalised $(n_1^2 + d_1^2 = n_2^2 + d_2^2 = 1)$, we have:

$$|\sin(\alpha)| = |\cos(90^o - \alpha)| = |(d_1, n_1) \cdot (-n_2, d_2)| = |-n_2 d_1 + d_2 n_1|$$

If we normalise $(1, p_1) \equiv \left(\frac{1}{\sqrt{1+p_1^2}}, \frac{p_1}{\sqrt{1+p_1^2}}\right)$ and $(1, p_2) \equiv \left(\frac{1}{\sqrt{1+p_2^2}}, \frac{p_2}{\sqrt{1+p_2^2}}\right)$, the result, in direct unnormalised terms is:

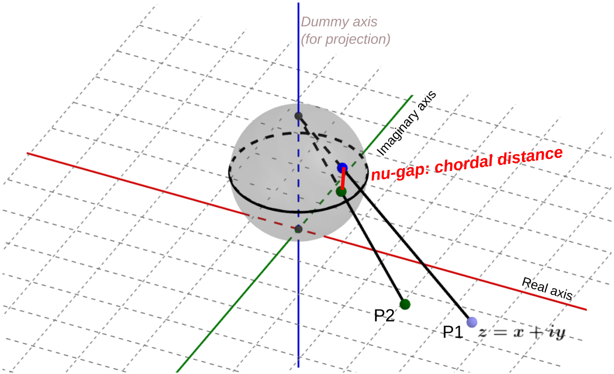
$$|\sin\alpha| = |\psi(p_1, p_2)| = \frac{|p_1 - p_2|}{\sqrt{1+p_1^2} \cdot \sqrt{1+p_2^2}}$$

Geometric interpretation (2): chordal distance



- The circumference here (diameter=1) is half the radius that the one in earlier slides (it had radius=1).
- This geometric construction is named **stereographic projection**.

Geometric intrepretation (2) complex case (SISO)



- Unit diameter circumference gets converted to spherical surface (Riemann Sphere)
- Horizontal line gets converted to complex plane, $P_1(j\omega)$, $P_2(j\omega)$.

*There is also a 'sine' interpretation, omitted for brevity.

Generalization to MIMO

We can define:

$$\Psi(P_1(s), P_2(s)) = \left\| (I + P_1^{\sim} P_1)^{1/2} (P_1 - P_2)(I + P_2^{\sim} P_2)^{1/2} \right\|_\infty$$

where $P^{\sim}(s) = P^T(-s)$... This is a common construct to extend to transfer function matrices the "conjugate transpose" concept over the imaginary axis (freq. response): $P^{\sim}(j\omega) = P^H(j\omega) = P^T(-j\omega)$.

From the normalised coprime factor representations $P_1 = N_1 D_1^{-1}$ y $P_2 = \tilde{D}_2^{-1}\tilde{N}_2$:

$$\Psi(P_1, P_2) = \| -\tilde{D}_2 N_1 + \tilde{N}_2 D_1 \|_\infty$$

Main result: ν-gap (Vinnicombe)

- Given two plants P_1 y P_2, we will define the ν-gap as:

$$\nu\text{-gap} = \begin{cases} \Psi(P_1, P_2) & \text{if certain technical conditions over unstable poles/zeros of } P_1,\ P_2 \text{ hold}^*, \\ 1 & \text{otherwise.} \end{cases}$$

- If $ncfmargin(P_1, K) > \nu\text{-gap}(P_1, P_2)$ then K stabilises P_2.

*Indeed, for robust stability via small gain (Δ_N, Δ_D) must be stable (well, it can be relaxed), so not only distance between frequency responses (determined by Ψ) is requiered, but extra stability requirements or, in the most general case, some Nyquist-related conditions in the original reference.

*Not too intuitive/important in robust control practice.
*But key for mathematical correctness, of course, see orig. references.

Conclusions

- The minimum distance between two plants understood as $(d_1 + \delta_d, n_1 + \delta_n) = q \cdot (d_2, n_2)$, minimum $\|(\delta_d, \delta_n)\|_2$ can be interpreted as a sine of the angle between the lines if $\|(d_1, n_1)\| = 1$, and as a **chordal distance** in stereographic projection.
- That "distance" between plants can be computed with the frequency response, jointly with formal conditions over unstable poles/zeros of $p_1(s)$, $p_2(s)$, δ_n, δ_d. This can be applied to MIMO $\mathcal{H}_\infty$ `ncfsyn` designs.
- It can also be computed from internal representation `ss(A,B,C,D)` (see the original Vinnicombe works).
- **Matlab:** `[~, ng]=gapmetric(1/(s+1),1/(s+3))`. Weights are usually needed to give a meaningful interpretation in actual applications.

Grosso modo, it shrinks uncertainty at large amplitudes.