ν-gap metric: intuitive understanding (real-valued, SISO case)

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Presentations in video:
http://personales.upv.es/asala/YT/V/**gapm1EN**.html, http://personales.upv.es/asala/YT/V/**gapm1bEN**.html

Introduction

Motivation: Coven a control loop `ncfmargin(P,K)` returns the margin to modelling error in *normalized coprime factor* representation $P = ND^{-1}$, $N^* \cdot N + D^* \cdot D = I$. The concept seems hard to understand and impractical, at least at first glance.

Objectives: Understand the geometric meaning of “normalised factorization error”, in a real, single-variable case for simplicity, to better apprehend the intuition for the general complex-matrix-valued case of Vinnicombe’s ν-gap.

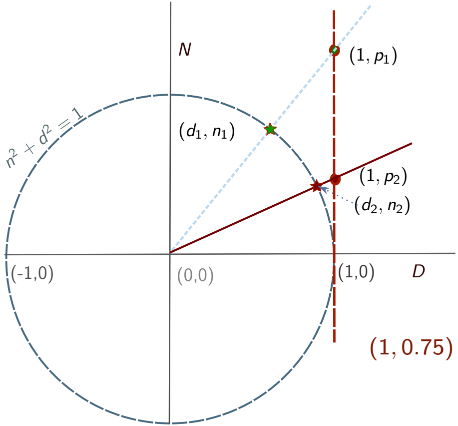
Uncertainty and Feedback, $\mathcal{H}_\infty$ Loop-Shaping and the ν-gap Metric, G. Vinnicombe (2000)

Real-valued geometrical interpretation (1)

We will represent $p = n/d$, $d \neq 0$, as $(d, n) \in \mathbb{R}^2 \sim \{(0, n)\}$.

Equivalent fractions $(d_a, n_a) \equiv (d_b, n_b)$ iff $n_a/d_a = n_b/d_b$.

Normalized factorisation iff $n^2 + d^2 = 1$.



Any $p_1 \in \mathbb{R}$, interpreted as fraction $p_1/1$, can be "projected" (normalized) to the circumference with center 0 radius 1, denoted as $\mathcal{B}$.

$$p_1/1 \text{ verifies } (1, p_1) \equiv (\overbrace{\frac{1}{\sqrt{1+p_1^2}}}^{d_1}, \overbrace{\frac{p_1}{\sqrt{1+p_1^2}}}^{n_1})$$

For instance:

$$(1, 0.75) \equiv (4, 3) \equiv (12, 9) \equiv (\tfrac{1}{\sqrt{1+0.75^2}}, \tfrac{0.75}{\sqrt{1+0.75^2}}) = (0.8, 0.6)$$

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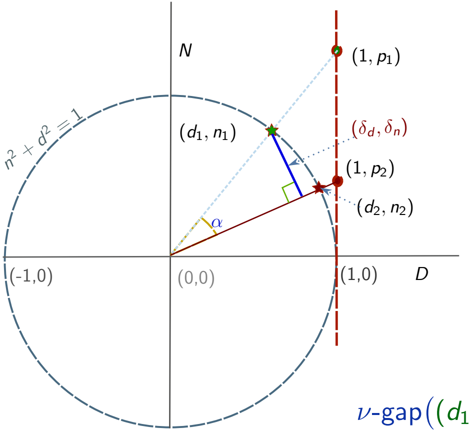

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Given $(d_1, n_1) \in \mathcal{B}$, the smallest (δ_d, δ_n) such that $(d_1 + \delta_d, n_1 + \delta_n) \equiv (d_2, n_2) \in \mathcal{B}$ is the difference between (d_1, n_1) and its orth. projection onto line $(\gamma d_2, \gamma n_2)$, $\gamma \in \mathbb{R}$. We will denote:

$$\nu\text{-gap}((d_1, n_1), (d_2, n_2)) := \|(\delta_d, \delta_n)\| = |\sin(\alpha)|$$

Normalised/unnormalised expressions of ν-gap

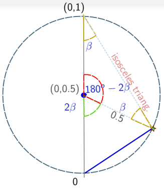
Scalar product of two norm-1 vectors is the *cosine* of the angle between them. Considering the perpendicular to (d_2, n_2), i.e., $(-n_2, d_2)$, then, assuming normalised $(n_1^2 + d_1^2 = n_2^2 + d_2^2 = 1)$, we have:

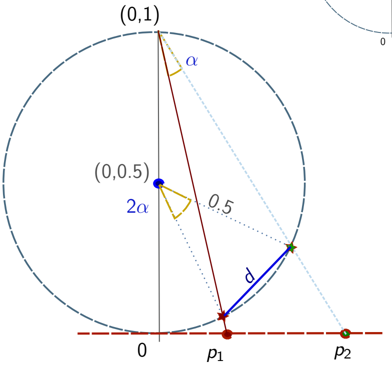
$$|\sin(\alpha)| = |\cos(90^o - \alpha)| = |(d_1, n_1) \cdot (-n_2, d_2)| = |-n_2 d_1 + d_2 n_1|$$

If we normalise $(1, p_1) \equiv \left(\frac{1}{\sqrt{1+p_1^2}}, \frac{p_1}{\sqrt{1+p_1^2}}\right)$ and $(1, p_2) \equiv \left(\frac{1}{\sqrt{1+p_2^2}}, \frac{p_2}{\sqrt{1+p_2^2}}\right)$, the result, in direct unnormalised terms is:

$$|\sin\alpha| = |\psi(p_1, p_2)| = \frac{|p_1 - p_2|}{\sqrt{1+p_1^2} \cdot \sqrt{1+p_2^2}}$$

Geometric interpretation (2): chordal distance





- The circumference here (diameter=1) is half the radius that the one in earlier slides (it had radius=1).

- The length of a chord that spans an angle β in a circumference of radius ρ is $d = 2\rho\sin(\beta/2)$.

$\Rightarrow$ In our case $\rho = 0.5$, $\beta = 2\alpha$, so $d = \sin\alpha$.

This geometric construction is named **stereographic projection**.

Conclusions

- Factorisations $p = d^{-1}n$ can be assimilated to a straight line $q \cdot (d, n)$ in the "graph" space in $\mathbb{R}^2$. The normalised factorisation is the intersection of the line with the unit radius circumference.
- The minimum distance between two normalised factorisations, so there $\exists q$ such that $(d_1 + \delta_d, n_1 + \delta_n) = q \cdot (d_2, n_2)$, can be shown to be the abs. value of the sine of the angle between associated lines.
- Said value is $|n_1 d_2 - d_1 n_2|$ (normalized), equiv. $|p_1 - p_2|/\sqrt{(1 + p_1^2)(1 + p_2^2)}$ (unnormalized).
- It can also be understood as a chordal distance in stereographic projection.
- The so-called ν-gap metric generalizes the idea to MIMO plants described as $P_1(s) = D_1^{-1}(s)N_1(s)$, $P_2(s) = D_2^{-1}(s)N_1(s)$.