Why do fingers cool before than the face when it is cold?

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Abstract

The purpose of this note is to show a physical application of vector calculus by using the heat equation.

In this note we will answer the question of the title under reasonable physical hypotheses and without considering medical reasons as blood pressure or sweating. Vector calculus is used in many fields of physical sciences: electromagnetism, gravitation, thermodynamics, fluid dynamics, ... (see, for example, [1]). Here, we will apply vector calculus to the heat equation in order to show a simple and intuitive physical fact. Throughout this note some scattered mathematical concepts shall appear, as the gradient, the chain rule, the space curves, the divergence theorem, the Leibniz integral rule, and the isoperimetric inequality.

We start with a brief introduction, the interested reader can consult [1, 2]. Let us consider a body which occupies a region $\Omega \subset \mathbb{R}^3$ with a temperature distribution. Let $T(x, y, z, t)$ be the temperature of the point $(x, y, z) \in \Omega$ at the time t. Thus, we can model the temperature as a mapping $T : \Omega \times \mathbb{R} \to \mathbb{R}$ differentiable enough. The colder regions warm up and the warmer regions cool, and therefore we can imagine that there is a heat flux from warmer areas to cooler ones. It is natural to assume that the magnitude of this flux is proportional to the spatial change of the temperature T. Fourier's Law relates the *heat flux*, $\mathbf{J}$, and the gradient of T:

$$\mathbf{J} = -k\nabla T = -k\left(\frac{\partial T}{\partial x}, \frac{\partial T}{\partial y}, \frac{\partial T}{\partial z}\right), \tag{1}$$

where k is a positive constant which depends on the material. For a given value of ∇T, when k increases, the norm of $\mathbf{J}$ increases, hence k measures the conductivity. For this reason, k is called the *thermal conductivity* of the material. The law of the conservation of the energy permits to prove the *heat equation*:

$$\frac{\partial T}{\partial t} = \kappa\left(\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial x^2}\right) = \kappa\nabla^2 T, \tag{2}$$

where $\kappa = k/(c\rho)$ is called the *thermal diffusivity*, the constant c is the *specific heat*, and ρ is the *density*. Till now everything is well known and it can be found in the aforementioned references.

The usual explanation for the minus sign in (1) is the following: *Heat flows from warm regions to colder regions, the vector ∇T points from cold regions to warmer regions. It is therefore logical that $\mathbf{J}$ and ∇T have opposite signs.* However, there is a more mathematical reason: Let $\mathbf{r} = \mathbf{r}(s)$ be a curve oriented in the direction of the flow, i.e., $\mathbf{r} : [a, b] \to \mathbb{R}^3$ with $\mathbf{r}'(s) = f(s)\mathbf{J}(\mathbf{r}(s))$ for some positive function f. For a fixed t, let us denote $T_t(x, y, z) = T(x, y, z, t)$. From (1) and the chain rule, we get $(T_t \circ \mathbf{r})'(s) = -kf(s)\|\nabla T(\mathbf{r}(s), t)\|^2 < 0$, thus the function $s \mapsto T(\mathbf{r}(s), t)$ is decreasing for a fixed t.

Divergence theorem is widely used in the applications of vector calculus. Is it possible to use this theorem together with (2) in order to deduce some interesting consequences?

Let S be the boundary of Ω. We will suppose that S is a closed orientable surface and let $\mathbf{N}$ be the outer normal unitary vector. From (1) and (2) we get

$$\iint_S \mathbf{J}\cdot \mathrm{d}\mathbf{S} = \iiint_\Omega \mathrm{div}\mathbf{J}\,\mathrm{d}v = -k\iiint_\Omega \nabla^2 T\,\mathrm{d}v = -c\rho \iiint_\Omega \frac{\partial T}{\partial t}\,\mathrm{d}v.$$

We will interchange the integral with the differentiation. This step is made by many textbooks without justifying it. Indeed, this step (known as *Leibniz integral rule*) is not obvious and it is a non trivial result of integration. Now

$$\iint_S \mathbf{J}\cdot \mathrm{d}\mathbf{S} = -c\rho\frac{\mathrm{d}}{\mathrm{d}t}\iiint_\Omega T\ \mathrm{d}v. \tag{3}$$

Can we write (3) in a shorter way? Using the average of the function T leads to

$$\iint_S \mathbf{J}\cdot \mathrm{d}\mathbf{S} = -c\rho\frac{\mathrm{d}}{\mathrm{d}t}(\mathrm{Volume}(\Omega)\overline{T}) = -c\rho\mathrm{Volume}(\Omega)\frac{\mathrm{d}\overline{T}}{\mathrm{d}t}, \tag{4}$$

where $\overline{T}$ denotes the average of $T:\Omega\to\mathbb{R}$. We can not simplify (4) anymore. However, under reasonable hypotheses we can simplify the left-hand of (4).

From now on, we shall suppose that $\|\mathbf{J}\|$ is a constant, say J, (which means that heat goes out of S uniformly) and $\mathbf{J}$ is normal to the surface S (which implies that $\mathbf{J}\cdot\mathbf{N}$ is maximum), Therefore $\mathbf{J}\cdot\mathbf{N} = J$, and thus $\iint_S \mathbf{J}\cdot \mathrm{d}\mathbf{S} = J\cdot \mathrm{Area}(S)$. Hence, by (4)

$$\frac{\mathrm{d}\overline{T}}{\mathrm{d}t} = -\frac{J}{c\rho}\frac{\mathrm{Area}(S)}{\mathrm{Volume}(\Omega)}. \tag{5}$$

This relation means that the bigger $\mathrm{Area}(S)/\mathrm{Volume}(\Omega)$ is, the quicker the body loses heat. What bodies do minimize this quotient for a fixed volume? This question is closely related with the three-dimensional *isoperimetric problem.*

The (planar) isoperimetric problem has been known from the time of antiquity. It can be stated as follows: *Find a closed plane curve of a given perimeter which encloses the greatest area.* Or its dual: *Find a closed plane curve which encloses a given area with* *the smaller perimeter*. The solution of these problems is a circle. These problems can be solved by means of the *isoperimetric inequality*: Let us denote the perimeter and enclosed area of a closed planar curve by L and A, respectively, then $L^2 \geq 4\pi A$ and the equality holds if and only if the curve is a circle (see [3] for some proofs and [4] for an elementary proof).

Isoperimetric inequality can be generalized in several ways. For one, it extends into higher dimensional spaces: for example, if A is the area of a surface and V is the volume of a three-dimensional body, then $A^3 \geq 36\pi V^2$, and the equality holds if and only if the body is a sphere. See [5, 6] for a pleasant explanation of the isoperimetric problem.

Hence the more spherical a body is, the smaller Area/Volumen is. Hence a spherical body loses heat slower than a non spherical body. Another way of understanding this result is when the area of a surface increases, the heat has more place to scape.

However, there is a mistake. From (5), it follows $\mathrm{d}\overline{T}/\mathrm{d}t < 0$, which implies that the considered body cools. Obviously, this is not true, bodies can warm up! In fact, $\mathbf{J} \cdot \mathbf{N} = \pm J$, depending of the orientation of $\mathbf{J}$. If we choose the plus sign, then $\mathbf{J}$ has the same orientation than $\mathbf{N}$, i.e., $\mathbf{J}$ is outer to S and therefore the heat goes out and body loses heat. If we choose the minus sign, then $\mathbf{J}$ is inner to S, heat goes in and body warms up.

References

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