Characterization of consistent completion of reciprocal comparison matrices

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What is the Analyty Hierarchy Process?

Decision making is becoming increasingly complex due to the large number of alternatives and multiple conflicting goals.

The Analytic Hierarchy Process (AHP) has been accepted as a leading multiattribute decision-aiding model.

Decisions arise in every type of application including

- Engineering
- Economy
- Medical management
- Emergency situations
- ...

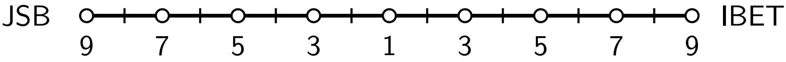
What is the Analyty Hierarchy Process?

Example: (Pairwise comparisons)

I want to submit a manuscript to one of the following two journals:

- Journal of Superb Mathematics
- International Bulletin of Excellent Theorems

What can I do?



1 Equal importance.

3 Moderate importance of one over another.

5 Strong importance.

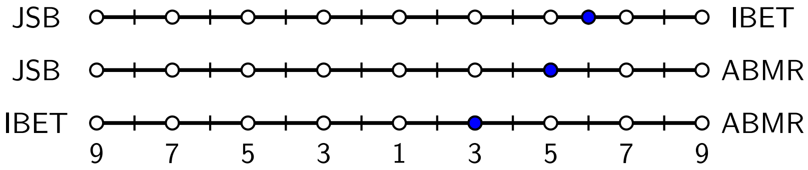
7 Very strong importance.

9 Extreme importance.

What is the Analyty Hierarchy Process?

Example: (Reciprocal matrices)

- Journal of Superb Mathematics
- International Bulletin of Excellent Theorems
- Annals of the Best Mathematical Results



	JSB	IBET	ABMR
JSB	1	6	5
IBET	1/6	1	3
ABMR	1/5	1/3	1

What is the Analyty Hierarchy Process?

Definition

A matrix $\mathbf{A} = (a_{ij})_{i,j=1}^{n} \in \mathbb{R}_{n,n}$ is reciprocal when $a_{ij} > 0$ and $a_{ij} = 1/a_{ji}$ for all $i, j \in \{1, \ldots, n\}$.

What is the Analyty Hierarchy Process?

Example: (Consistent matrices)

$$
\begin{array}{c} \\ \text{JSB} \\ \text{IBET} \\ \text{ABMR} \end{array}
\begin{array}{c} \begin{array}{ccc} \text{JSB} & \text{IBET} & \text{ABMR} \end{array} \\ \begin{bmatrix} 1 & 6 & 5 \\ 1/6 & 1 & 3 \\ 1/5 & 1/3 & 1 \end{bmatrix} \end{array}
$$

- I prefer JSB 6 times over IBET.
- I prefer IBET 3 times over ABMR.
- I prefer JSB 5 times over ABMR.

This is not “rational”. I had to prefer JSB 6×3 times over ABMR!

$$6 \times 3 \neq 5$$

What is the Analyty Hierarchy Process?

Definition

A matrix $\mathbf{A} = (a_{ij})_{i,j=1}^{n} \in \mathbb{R}_{n,n}$ is consistent when $a_{ij} > 0$ and $a_{ij}a_{jk} = a_{ik}$ for all $i, j, k \in \{1, \ldots, n\}$.

Theorem

Let $\mathbf{A} \in \mathbb{R}_{n,n}$ be positive.

- *If $\mathbf{A}$ is consistent, then $\mathbf{A}$ is reciprocal.*
- *If $\mathbf{A}$ is reciprocal and $n = 2$, then $\mathbf{A}$ is consistent.*

What is the Analyty Hierarchy Process?

T. Saaty (1970) proposed a measure of the inconstency of a reciprocal matrix $\mathbf{A}$ based on Perron theory.

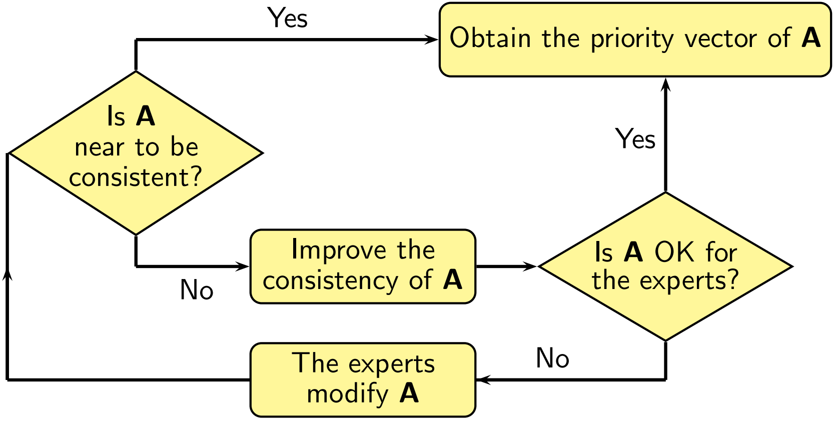
Theorem (O. Perron 1907)

Let $\mathbf{A} \in \mathbb{R}_{n,n}$ *be positive. Let* $\rho(\mathbf{A}) := \max\{|\lambda| : \lambda \text{ is an eigenvalue of } \mathbf{A}\}$*. The following statements hold:*

- $\rho(\mathbf{A})$ *is an eigenvalue of* $\mathbf{A}$*, and there is a positive vector* $\mathbf{x}$ *such that* $\mathbf{Ax} = \rho(\mathbf{A})\mathbf{A}$*.*
- *If* λ *is an eigenvalue of* $\mathbf{A}$*, then* $|\lambda| < \rho(\mathbf{A})$*.*
- *The eigenspace corresponding to* $\rho(\mathbf{A})$ *has dimension 1.*

If $\mathbf{A}$ is a matrix “near” to be consistent, the eigenvector corresponding to $\rho(\mathbf{A})$ is called the priority vector.

What is the Analyty Hierarchy Process?



What is the Analyty Hierarchy Process?

Example: (two levels)

I want to submit a paper to JSB, IBET, or ABMR, but considering several criteria: Quickness of publication, Quality of typesetting, Quality of refereeing, Prestige of the journal.

What is the Analyty Hierarchy Process?

Example: (two levels)
I want to submit a paper to JSB, IBET, or ABMR, but considering several criteria: Quickness of publication, Quality of typesetting, Quality of refereeing, Prestige of the journal.
First step: Determine the relative importance of the criteria:

$$\begin{array}{c} \text{Quickness of publication} \\ \text{Quality of typesetting} \\ \text{Quality of refereeing} \\ \text{Prestige of the journal} \end{array} \begin{bmatrix} 1 & 7 & 2 & 1/3 \\ 1/7 & 1 & 1/5 & 1/9 \\ 1/2 & 5 & 1 & 1/2 \\ 3 & 9 & 2 & 1 \end{bmatrix} = \mathbf{A}$$

If **A** is near to be consistent, obtain the priority vector

$$\mathbf{w} = (w_1, w_2, w_3, w_4), \qquad w_1 + w_2 + w_3 + w_4 = 1.$$

What is the Analyty Hierarchy Process?

Example: (two levels)
I want to submit a paper to JSB, IBET, or ABMR, but considering several criteria: Quickness of publication, Quality of typesetting, Quality of refereeing, Prestige of the journal

Second step: For each criteria, order the alternatives:

Criteria (weight)	JSB	IBET	ABMR	
Quickness of publication (w_1)	v_{11}	v_{12}	v_{13}	$\to \mathbf{v}_1 \in \mathbb{R}^3$
Quality of Typesetting (w_2)	v_{21}	v_{22}	v_{23}	$\to \mathbf{v}_2 \in \mathbb{R}^3$
Quality of refereeing (w_3)	v_{31}	v_{32}	v_{33}	$\to \mathbf{v}_3 \in \mathbb{R}^3$
Prestige of the journal (w_4)	v_{41}	v_{42}	v_{43}	$\to \mathbf{v}_4 \in \mathbb{R}^3$

Priority vector: $w_1\mathbf{v}_1 + w_2\mathbf{v}_2 + w_3\mathbf{v}_3 + w_4\mathbf{v}_4$

Statement of the problem

Sometimes, the appearing comparison matrices can be incomplete.

- Some experts may not be completely familiar with one or more of the elements.
- It is possible that there exist conflicts of interests.
- Some experts may not want to express their opinion.
- Some data may be lost.
- ...

Statement of the problem

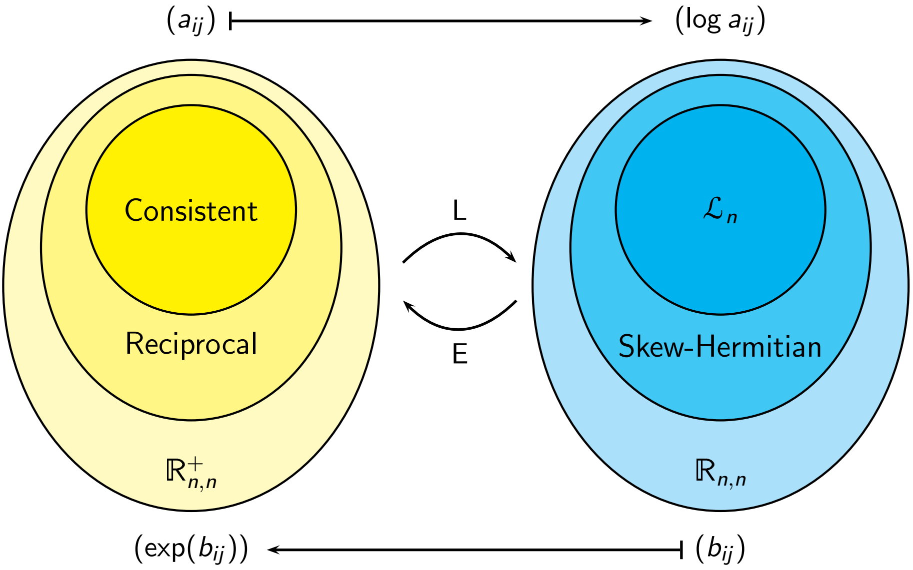
Sometimes, the appearing reciprocal matrices can be incomplete.

$$\mathbf{A} = \begin{bmatrix} 1 & 2 & 3 & \star \\ 1/2 & 1 & 3 & 4 \\ 1/3 & 1/3 & 1 & \star \\ \star & 1/4 & \star & 1 \end{bmatrix}$$

Problem

When $\mathbf{A}$ *can be completed to be consistent?*
In case that $\mathbf{A}$ *can be completed, find all consistent completations.*

Preliminaries – Linearization



Notations – Statement of the theorem

Example:

$$\begin{array}{ccc} \mathbf{A} & & L(\mathbf{A}) \\ \begin{bmatrix} 1 & 2 & 3 & \star \\ 1/2 & 1 & 3 & 4 \\ 1/3 & 1/3 & 1 & \star \\ \star & 1/4 & \star & 1 \end{bmatrix} & \Longleftrightarrow & \begin{bmatrix} 0 & \log 2 & \log 3 & \star \\ -\log 2 & 0 & \log 3 & \log 4 \\ -\log 3 & -\log 3 & 0 & \star \\ \star & -\log 4 & \star & 0 \end{bmatrix} \end{array}$$

Any skew-Hermitiam completation of $L(\mathbf{A})$ is of the form

$$\mathbf{C}(\lambda, \mu) = \begin{bmatrix} 0 & \log 2 & \log 3 & \lambda \\ -\log 2 & 0 & \log 3 & \log 4 \\ -\log 3 & -\log 3 & 0 & \mu \\ -\lambda & -\log 4 & -\mu & 0 \end{bmatrix}$$

Notations – Statement of the theorem

Example: $\mathbf{A} = \begin{bmatrix} 1 & 2 & 3 & \star \\ 1/2 & 1 & 3 & 4 \\ 1/3 & 1/3 & 1 & \star \\ \star & 1/4 & \star & 1 \end{bmatrix}$

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$$+ \lambda \underbrace{\begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{bmatrix}}_{\mathbf{e}_1\mathbf{e}_4^T - \mathbf{e}_4\mathbf{e}_1^T} + \mu \underbrace{\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{bmatrix}}_{\mathbf{e}_3\mathbf{e}_4^T - \mathbf{e}_4\mathbf{e}_3^T} \quad \text{entries to fill}$$

Notations – Statement of the theorem

Notations

$$N_n = \{(i,j) : 1 \le i < j \le n\}, \qquad \mathbf{B}_{ij} = \mathbf{e}_i \mathbf{e}_j^T - \mathbf{e}_j \mathbf{e}_i^T, \ (i,j) \in N_n$$

Example: The skew-Hermitian completation considered can be written

$$\mathbf{C}(\lambda, \mu) = \mathbf{C}_0 + \lambda \mathbf{B}_{14} + \mu \mathbf{B}_{34},$$

where

$$\mathbf{C}_0 = \sum_{(i,j) \in N_4 \setminus \{(1,4),(3,4)\}} \rho_{ij} \mathbf{B}_{ij},$$

and $\rho_{ij} \in \mathbb{R}$ easily determined from the incomplete matrix **A**.

- $\mathbf{C}_0$ is the incomplete skew-Hermitian matrix to be completed.
- $(1,4), (3,4)$ are the void positions that must be filled.

Consistent completation of a reciprocal matrix

The unspecified entries of $\mathbf{A}$ will be located at the indices belonging to I.

$$\mathbf{A} = \begin{bmatrix} 1 & 2 & 3 & \star \\ 1/2 & 1 & 3 & 4 \\ 1/3 & 1/3 & 1 & \star \\ \star & 1/4 & \star & 1 \end{bmatrix}$$

I J

Theorem

Let $1 \leq i_1, j_1, \ldots, i_k, j_k \leq n$ be indices such that $i_r < j_r$ for $r = 1, \ldots, k$. Denote $I = \{(i_1, j_1), \ldots, (i_k, j_k)\}$ and $J = N_n \setminus I$. Let $\mathbf{C}_0 = \sum_{(i,j)\in J} \rho_{ij}\mathbf{B}_{ij}$. The following statements are equivalent

(i) *There exist $\lambda_1, \ldots, \lambda_k \in \mathbb{R}$ such that $\mathbf{C}_0 + \sum_{r=1}^{k} \lambda_r \mathbf{B}_{i_r j_r} \in \mathcal{L}_n$.*

(ii) *There exists $\mathbf{w} = (w_1, \ldots, w_n)^T \in \mathbb{R}^n$ such that $\rho_{pq} = w_p - w_q$ for any $(p, q) \in J$.*

Furthermore, in the case that the statements hold, then

$$\lambda_r = w_{i_r} - w_{j_r}, \qquad \forall\ r \in \{1, \ldots, k\}.$$

Consistent completation of a reciprocal matrix

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We have reduced the completation problem to study the linear system occurring in item (ii).

- The coefficients of this system are 0, -1, 1.
- A row has only two non-zero entries (sparsity).

Example

$$\mathbf{A} = \begin{bmatrix} 1 & 2 & 3 & \star \\ 1/2 & 1 & 3 & 4 \\ 1/3 & 1/3 & 1 & \star \\ \star & 1/4 & \star & 1 \end{bmatrix}$$

If **A** had a consistent completation, then there would exist $\mathbf{w} = (w_1, w_2, w_3, w_4)^T \in \mathbb{R}^4$ such that

$$\begin{aligned} \log 2 &= w_1 - w_2 \\ \log 3 &= w_1 - w_3 \\ \log 3 &= w_2 - w_3 \\ \log 4 &= w_2 - w_4 \end{aligned}$$

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This linear system has no solution

Another example

$$\mathbf{A} = \begin{bmatrix} 1 & \star & 1/3 \\ \star & 1 & 2/3 \\ 3 & 3/2 & 1 \end{bmatrix}$$

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By taking logarithms, we construct

$$\mathbf{C}_0 = \begin{bmatrix} 0 & 0 & -\log 3 \\ 0 & 0 & \log 2 - \log 3 \\ \log 3 & \log 3 - \log 2 & 0 \end{bmatrix}.$$

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If there were a consistent completation, then there would exist $\mathbf{w} = (w_1, w_2, w_3)^T \in \mathbb{R}^3$ such that

$$-\log 3 = w_1 - w_3 \qquad \log 2 - \log 3 = w_2 - w_3$$

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This linear system is solvable. Thus, exists a consistent completation of **A**.

Another example (continuation)

$$\mathbf{A} = \begin{bmatrix} 1 & \star & 1/3 \\ \star & 1 & 2/3 \\ 3 & 3/2 & 1 \end{bmatrix} \quad \mathbf{C}_0 = \begin{bmatrix} 0 & 0 & -\log 3 \\ 0 & 0 & \log 2 - \log 3 \\ \log 3 & \log 3 - \log 2 & 0 \end{bmatrix}$$

The solution of the prior system is

$$w_1 = -\log 3 + \alpha, \quad w_2 = \log 2 - \log 3 + \alpha, \quad w_3 = \alpha, \quad \alpha \in \mathbb{R}.$$

If $\mathbf{X}$ is any consistent completation of $\mathbf{A}$, then

$$L(\mathbf{X}) = \mathbf{C}_0 + (w_1 - w_2)\mathbf{B}_{12} = \mathbf{C}_0 - \log 2 \cdot \mathbf{B}_{12}.$$

There is a unique consistent completation of $\mathbf{A}$:

$$\mathbf{X} = \begin{bmatrix} 1 & 1/2 & 1/3 \\ 2 & 1 & 2/3 \\ 3 & 3/2 & 1 \end{bmatrix}$$

Connection with the graph theory

$$\begin{bmatrix} 1 & 2 & 3 & \star \\ 1/2 & 1 & 3 & 4 \\ 1/3 & 1/3 & 1 & \star \\ \star & 1/4 & \star & 1 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & -1 & 0 & 0 \\ 1 & 0 & -1 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 1 & 0 & -1 \end{bmatrix} \begin{bmatrix} w_1 \\ w_2 \\ w_3 \\ w_4 \end{bmatrix} = \begin{bmatrix} \log 2 \\ \log 3 \\ \log 3 \\ \log 4 \end{bmatrix}$$

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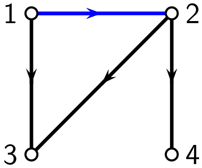
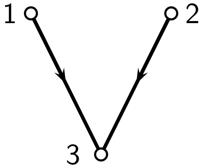
The transpose of the matrices are the incidence matrix of the graphs





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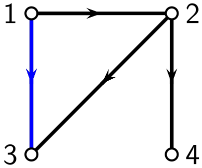

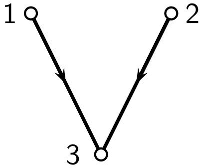

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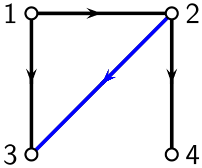

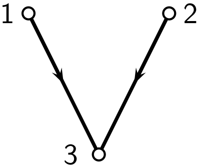

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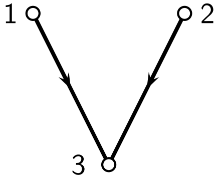
Notations

$$\mathbf{A} = \begin{bmatrix} 1 & \star & 1/3 \\ \star & 1 & 2/3 \\ 3 & 3/2 & 1 \end{bmatrix}$$

$$\underbrace{\begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \end{bmatrix}}_{:=M_{\mathbf{A}}^T} \underbrace{\begin{bmatrix} w_1 \\ w_2 \\ w_3 \end{bmatrix}}_{:=\mathbf{w}} = \underbrace{\begin{bmatrix} -\log 3 \\ \log 2 - \log 3 \end{bmatrix}}_{:=\mathbf{b}_{\mathbf{A}}}$$



$G_{\mathbf{A}}$

Some consequences

Theorem

If $\mathbf{A}$ *is an incomplete reciprocal matrix, then* $\mathbf{A}$ *can be completed to be a consistent matrix if and only if the system* $M_{\mathbf{A}}^T \mathbf{w} = \mathbf{b}_{\mathbf{A}}$ *is consistent.*

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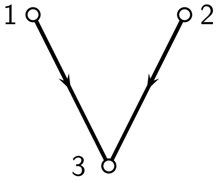
Theorem

Let $\mathbf{A} \in \mathbb{R}_{n,n}$ *be a reciprocal incomplete matrix and* $2k$ *be the number of void entries (located up and down the main diagonal of* $\mathbf{A}$*). If* $G_{\mathbf{A}}$ *has* p *connected components and* $2k \geq n^2 - 3n + 2p$*, then* $\mathbf{A}$ *can be completed to be consistent.*

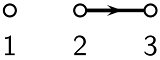
$$\mathbf{A} = \begin{bmatrix} 1 & \star & 1/3 \\ \star & 1 & 2/3 \\ 3 & 3/2 & 1 \end{bmatrix}$$



$$n = 3 \quad k = 1 \quad p = 1$$

Example (disconnected graph)

$$\mathbf{A} = \begin{bmatrix} 1 & \star & \star \\ \star & 1 & a \\ \star & 1/a & 1 \end{bmatrix}$$



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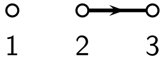

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Any consistent completation of **A** is of the following form

$$\mathbf{X} = \begin{bmatrix} 1 & b & ab \\ 1/b & 1 & a \\ 1/(ab) & 1/a & 1 \end{bmatrix}.$$

The completation of **A** is not unique.

A result on the uniqueness

Theorem

Let $\mathbf{A} \in \mathbb{R}_{n,n}$ *be a reciprocal incomplete matrix. If* $G_{\mathbf{A}}$ *is connected and exists a consistent completation of* $\mathbf{A}$*, then this completation is unique.*

Let us recall

Theorem

If $\mathbf{A}$ *is an incomplete reciprocal matrix, then* $\mathbf{A}$ *can be completed to be a consistent matrix if and only if the system* $M_{\mathbf{A}}^T \mathbf{w} = \mathbf{b}_{\mathbf{A}}$ *is consistent.*

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More matrix algebra and graph theory

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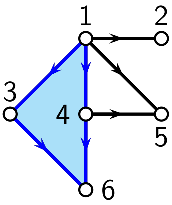
The null space of $M_{\mathbf{A}}$ corresponds to the cycles of $G_{\mathbf{A}}$



$$\begin{bmatrix} 1 & 1 & 1 & 1 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & -1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 & -1 \end{bmatrix} \begin{bmatrix} 0 & 0 \\ -1 & -1 \\ 1 & 0 \\ 0 & 1 \\ -1 & -1 \\ 0 & -1 \\ 1 & 1 \end{bmatrix} = \mathbf{0}$$

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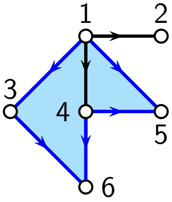

$$\begin{bmatrix} 1 & 1 & 1 & 1 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & -1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 & -1 \end{bmatrix} \begin{bmatrix} 0 & 0 \\ -1 & -1 \\ 1 & 0 \\ 0 & 1 \\ -1 & -1 \\ 0 & -1 \\ 1 & 1 \end{bmatrix} = \mathbf{0}$$

More matrix algebra and graph theory

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Sometimes we can find this null space very easily...

More matrix algebra and graph theory

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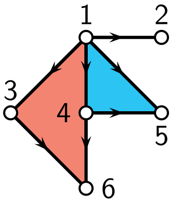
Theorem

Let G *be a planar graph and* M *its incidence matrix. If* $\mathbf{x}_1, \ldots, \mathbf{x}_f$ *correspond to the faces, then* $\{\mathbf{x}_1, \ldots, \mathbf{x}_f\}$ *is a basis of* $\mathcal{N}(M)$.



$[0\ 0\ -1\ 1\ 0\ -1\ 0]^T$

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More matrix algebra and graph theory

Theorem

- *If* $\mathbf{A}$ *is an incomplete reciprocal matrix, then* $\mathbf{A}$ *can be completed to be a consistent matrix if and only if* $\mathbf{b}_{\mathbf{A}}^T \mathbf{x} = 0$ *for any* $\mathbf{x} \in \mathcal{N}(M_{\mathbf{A}})$.
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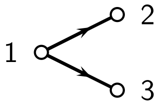
Corollary

Let $\mathbf{A}$ *be an incomplete reciprocal matrix. If* $G_{\mathbf{A}}$ *is planar and has no faces, then there exists a consistent completation of* $\mathbf{A}$.



Many thanks for your kind attention.

These slides are avalaible at

http://personales.upv.es/jbenitez/investigacion.html