The spectrum of matrices depending on two idempotents

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Abstract

Let P and Q be two complex matrices satisfying $P^2 = P$ and $Q^2 = Q$. If a, b are nonzero complex numbers such that $aP + bQ$ is diagonalizable, we relation the spectrum of $aP + bQ$ with the spectrum of $P - Q$, PQ, PQP and $PQ - QP$.

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Let $\mathbb{C}^{m\times n}$ denote the set of complex $m \times n$ matrices. A matrix P is said to be idempotent when $P^2 = P$. Furthermore, a matrix P is said to be an orthogonal projector when $P^2 = P = P^*$. The symbol I_n will stand for the identity matrix of order n. For $A \in \mathbb{C}^{m\times n}$, the symbols $\mathrm{rk}(A)$ and $\sigma(A)$ will denote the rank of A and the spectrum of A.

Recently, many properties concerning two orthogonal projectors have been deduced (see, e.g., [6, 8]). If we remove the hermitancy property (i.e., if we study expressions depending on two idempotents), the study becomes harder. However, some results can be found in the literature (see [9, 13, 14] and references therein). In this paper, we shall study the spectrum of several matrices depending on two idempotents. A related result was deduced in [6, Theorem 2.8].

When the idempotent matrices $P, Q \in \mathbb{C}^{n\times n}$ commute, the study of the spectrum of $aP + bQ$ for $a, b \in \mathbb{C} \setminus \{0\}$ is easy, and we can make it by using the following two well-known results:

a) Every idempotent matrix A is diagonalizable and $\sigma(A) \subset \{0, 1\}$ [16, Theorem 4.1].

b) Two diagonalizable matrices commute if and only if they are simultaneously diagonalizable [10, Theorem 1.3.19].

Theorem 1. *Let $P, Q \in \mathbb{C}^{n\times n}$ be two idempotent matrices such that $PQ = QP$ and let $a, b \in \mathbb{C} \setminus \{0\}$. Then $\sigma(aP + bQ) \subset \{0, a, b, a + b\}$.*

Proof. Let $x = \mathrm{rk}(PQ)$, $y = \mathrm{rk}(P)$, and $z = \mathrm{rk}(Q)$. There exist a nonsingular $S \in \mathbb{C}^{n\times n}$ such that $P = S(I_x \oplus I_{y-x} \oplus 0 \oplus 0)S^{-1}$ and $Q = S(I_x \oplus 0 \oplus I_{z-x} \oplus 0)S^{-1}$. It is clear that

$$aP + bQ = S((a + b)I_x \oplus aI_{y-x} \oplus bI_{z-x} \oplus 0)S^{-1}.$$

This concludes the proof. □

In the rest of the paper, we will relation $\sigma(aP + bQ)$ with $\sigma(P - Q)$, $\sigma(PQ)$, $\sigma(PQP)$, and $\sigma(PQ - QP)$ under the assumption that $aP + bQ$ is diagonalizable. Let us recall that the subset of $\mathbb{C}^{n\times n}$ composed of diagonalizable matrices is dense in $\mathbb{C}^{n\times n}$ and the Lebesgue measure of the subset composed of non diagonalizable matrices is 0. We need the following rather technical lemma to study the aforementioned spectrums.

Lemma 1. *Let $P, Q \in \mathbb{C}^{n\times n}$ be two idempotent matrices such that $PQ \neq QP$ and let $a, b \in \mathbb{C}\setminus\{0\}$ such that $aP + bQ$ is diagonalizable.*

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(i) *There exist a nonsingular* $S \in \mathbb{C}^{n\times n}$ *and idempotents* $P_0, \ldots, P_k, Q_0, \ldots Q_k$ *such that* $P_i, Q_i \in \mathbb{C}^{m_i \times m_i}$ *for* $i = 0, \ldots, k$,

$$P = S\left((\oplus_{i=1}^k P_i) \oplus P_0\right) S^{-1}, \qquad Q = S\left((\oplus_{i=1}^k Q_i) \oplus Q_0\right) S^{-1}, \tag{1}$$

$P_0Q_0 = Q_0P_0$*, and* $P_iQ_i \neq Q_iP_i$ *for* $i = 1, \ldots, k$.

(ii) *There exist pairwise distinct complex numbers* $\mu_1, \nu_1, \ldots, \mu_k, \nu_k$ *such that*

$$a + b = \mu_i + \nu_i, \quad \sigma(aP_i + bQ_i) = \{\mu_i, \nu_i\}, \quad ab(P_i - Q_i)^2 = \mu_i\nu_i I_{m_i} \tag{2}$$

for $i = 1, \ldots, k$.

(iii) *For* $i = 1, \ldots, k$*, there exist nonsingular matrices* S_i *such that*

$$P_i = S_i \begin{bmatrix} I_{x_i} & 0 \\ 0 & 0 \end{bmatrix} S_i^{-1}, \qquad Q_i = S_i \begin{bmatrix} A_i & B_i \\ C_i & D_i \end{bmatrix} S_i^{-1}, \tag{3}$$

where $x_i = \mathrm{rk}(P_i)$, $A_i \in \mathbb{C}^{x_i \times x_i}$, $D_i \in \mathbb{C}^{(m_i - x_i)\times(m_i - x_i)}$*, and*

$$A_i = \left(1 - \frac{\mu_i\nu_i}{ab}\right) I_{x_i}, \tag{4}$$

$$B_iC_i = \frac{\mu_i\nu_i}{ab}\left(1 - \frac{\mu_i\nu_i}{ab}\right) I_{x_i}, \tag{5}$$

$$C_iB_i = \frac{\mu_i\nu_i}{ab}\left(1 - \frac{\mu_i\nu_i}{ab}\right) I_{m_i - x_i}, \tag{6}$$

and

$$D_i = \frac{\mu_i\nu_i}{ab} I_{m_i - x_i}. \tag{7}$$

Proof. Let us denote

$$X = aP + bQ. \tag{8}$$

Since $XP - PX = b(QP - PQ)$ and $PQ \neq QP$ we obtain $PX \neq XP$. Expression (8) can be equivalently written as

$$Q = \alpha P + \beta X, \qquad \alpha = -\frac{a}{b}, \quad \beta = \frac{1}{b}. \tag{9}$$

Idempotency of Q leads to $(\alpha P + \beta X)^2 = \alpha P + \beta X$, which, in view of $P^2 = P$, simplifies to

$$(\alpha^2 - \alpha)P + \beta^2 X^2 + \alpha\beta(PX + XP) = \beta X. \tag{10}$$

Since X is diagonalizable, there exists a nonsingular $S \in \mathbb{C}^{n\times n}$ such that

$$X = S(\lambda_1 I_{p_1} \oplus \cdots \oplus \lambda_m I_{p_m})S^{-1} \tag{11}$$

with $\lambda_i \neq \lambda_j$ whenever $i \neq j$ and $p_1 + \cdots + p_m = n$. Let us represent P as

$$P = S \begin{bmatrix} P_{11} & \cdots & P_{1m} \\ \vdots & \ddots & \vdots \\ P_{m1} & \cdots & P_{mm} \end{bmatrix} S^{-1}, \tag{12}$$

with $P_{ii} \in \mathbb{C}^{p_i \times p_i}$ for $i = 1, \ldots, m$. We get

$$XP = S \begin{bmatrix} \lambda_1 P_{11} & \cdots & \lambda_1 P_{1m} \\ \vdots & \ddots & \vdots \\ \lambda_m P_{m1} & \cdots & \lambda_m P_{mm} \end{bmatrix} S^{-1}, \quad PX = S \begin{bmatrix} \lambda_1 P_{11} & \cdots & \lambda_m P_{1m} \\ \vdots & \ddots & \vdots \\ \lambda_1 P_{m1} & \cdots & \lambda_m P_{mm} \end{bmatrix} S^{-1}. \tag{13}$$

Relation (10) and representations (11), (12), and (13) imply that

$$[\alpha - 1 + \beta(\lambda_r + \lambda_s)]\, P_{rs} = 0 \tag{14}$$

holds for every $r, s \in \{1, \ldots, m\}$ such that $r \neq s$.

In view of (13) and $XP \neq PX$, we deduce that there exist $i, j \in \{1, \ldots, m\}$ such that $i \neq j$ and $\lambda_i P_{ij} \neq \lambda_j P_{ij}$, and thus $P_{ij} \neq 0$. We have from (14) the relationship $\alpha + \beta(\lambda_i + \lambda_j) = 1$ for $i \neq j$. Using the second and the third relations of (9) we have

$$a + b = \lambda_i + \lambda_j. \tag{15}$$

with $i \neq j$.

We rearrange the subindexes in such a way that $i = 1$ and $j = 2$. Assume that there exists $r \in \{3, \ldots, m\}$ such that $a + b = \lambda_1 + \lambda_r$. Combining this last equality with (15) leads to $\lambda_1 + \lambda_2 = \lambda_1 + \lambda_r$, which yields $\lambda_2 = \lambda_r$, a contradiction. Thus $\lambda_1 + \lambda_r \neq a + b$ for any $r \in \{3, \ldots, m\}$. Using (14), the second and the third relations of (9) leads to $P_{1r} = 0$ for all $r \in \{3, \ldots, m\}$. And a symmetric reasoning permits to obtain $P_{2r} = 0$, $P_{r1} = 0$, and $P_{r2} = 0$ for all $r \in \{3, \ldots, m\}$. Whence P can be written as

$$P = S\left(P_1 \oplus \widetilde{P}\right) S^{-1}, \qquad P_1 = \begin{bmatrix} P_{11} & P_{12} \\ P_{21} & P_{22} \end{bmatrix}, \quad P_{11} \in \mathbb{C}^{p_1 \times p_1}, \; P_{22} \in \mathbb{C}^{p_2 \times p_2} \tag{16}$$

for some square matrix $\widetilde{P}$ of suitable size. Furthermore, P_1 and $\widetilde{P}$ are idempotent because P is idempotent. From now on, we shall denote $\mu_1 = \lambda_1$, $\nu_1 = \lambda_2$, $r_1 = p_1$, and $s_1 = p_2$.

From (11), (16), and $Q = \alpha P + \beta X$, we obtain

$$X = S\left((\mu_1 I_{r_1} \oplus \nu_1 I_{s_1}) \oplus \Lambda_2\right) S^{-1}, \qquad Q = S\left(Q_1 \oplus \widetilde{Q}\right) S^{-1},$$

where Λ_2 is diagonal and $Q_1 \in \mathbb{C}^{(r_1+s_1)\times(r_1+s_1)}$. Observe that $aP_1 + bQ_1 = \mu_1 I_{r_1} \oplus \nu_1 I_{s_1}$.

If $\widetilde{P}\widetilde{Q} = \widetilde{Q}\widetilde{P}$, then it is enough to take $P_0 = \widetilde{P}$ and $Q_0 = \widetilde{Q}$ to prove (i) and the two first equalities of (2). Assume $\widetilde{P}\widetilde{Q} \neq \widetilde{Q}\widetilde{P}$. Let us observe that we have partitioned the matrices P, Q, and X as follows

$$X = S\begin{bmatrix} \mu_1 I_{r_1} \oplus \nu_1 I_{s_1} & 0 \\ 0 & \Lambda_2 \end{bmatrix} S^{-1}, \quad P = S\begin{bmatrix} P_1 & 0 \\ 0 & \widetilde{P} \end{bmatrix} S^{-1}, \quad Q = S\begin{bmatrix} Q_1 & 0 \\ 0 & \widetilde{Q} \end{bmatrix} S^{-1}.$$

From (8) we obtain $\Lambda_2 = a\widetilde{P} + b\widetilde{Q}$. Since $\Lambda_2\widetilde{P} - \widetilde{P}\Lambda_2 = b(\widetilde{Q}\widetilde{P} - \widetilde{P}\widetilde{Q}) \neq 0$ (recall that $\widetilde{P}$ and $\widetilde{Q}$ are idempotents) and by having in mind that Λ_2 is a diagonal matrix, (in fact, Λ_2 is obtained by removing in the expression (11) the two first summands, getting $\Lambda_2 = \lambda_3 I_{p_3} \oplus \cdots \oplus \lambda_{p_m} I_{p_m}$) by doing the same procedure as before, we can write

$$\Lambda_2 = \mu_2 I_{r_2} \oplus \nu_2 I_{s_2} \oplus \Lambda_3, \qquad \widetilde{P} = P_2 \oplus \widetilde{\widetilde{P}}, \qquad P_2 = \begin{bmatrix} \widetilde{P}_{11} & \widetilde{P}_{12} \\ \widetilde{P}_{21} & \widetilde{P}_{22} \end{bmatrix}, \tag{17}$$

where $\widetilde{P}_{11} \in \mathbb{C}^{r_2 \times r_2}, \widetilde{P}_{22} \in \mathbb{C}^{s_2 \times s_2}$. Since $\widetilde{Q} = \frac{1}{b}(\Lambda_2 - a\widetilde{P})$ from (17), we can partition $\widetilde{Q} = Q_2 \oplus \widetilde{\widetilde{Q}}$, where the size of Q_2 is the same as the sizes of $\mu_2 I_{r_2} \oplus \nu_2 I_{s_2}$ and P_2. In addition, we obtain $a + b = \mu_2 + \nu_2$, the cardinal of the set $\{\mu_1, \nu_2, \mu_2, \nu_2\}$ is exactly four (because in the representation (11), all λs are pairwise distinct). Observe that $aP_2 + bQ_2 = \mu_2 I_{r_2} \oplus \nu_2 I_{s_2}$. Now, an exhaustion process permits to prove (i), the first and the second relations of (2).

We shall prove the last equality of (2). Let us denote $X_i = aP_i + bQ_i$ for $i = 1, \ldots k$, and observe that there exists $r_i, s_i \in \{1, \ldots, m_i\}$ such that

$$X_i = \mu_i I_{r_i} \oplus \nu_i I_{s_i} \tag{18}$$

and $r_i+s_i=m_i$. From (18) we get $0=(X_i-\mu_i I_{m_i})(X_i-\nu_i I_{m_i})$, which, in view of $a+b=\mu_i+\nu_i$, simplifies to $X_i^2-(a+b)X_i+\mu_i\nu_i I_{m_i}=0$. Using $X_i=aP_i+bQ_i$ and the idempotency of P_i and Q_i leads to

$$\begin{aligned} 0 &= (aP_i+bQ_i)^2-(a+b)(aP_i+bQ_i)+\mu_i\nu_i I_{m_i} \\ &= ab(P_iQ_i+Q_iP_i-P_i-Q_i)+\mu_i\nu_i I_{m_i} \\ &= -ab(P_i-Q_i)^2+\mu_i\nu_i I_{m_i}. \end{aligned}$$

We shall prove (iii). Let us fix $i\in\{1,\ldots,k\}$. Matrices P_i and Q_i are idempotent because P and Q are idempotent. It is known that every idempotent matrix is diagonalizable [16, Theorem 4.1], and thus there exists a nonsingular matrix $S_i\in\mathbb{C}^{m_i\times m_i}$ such that $P_i=S(I_{x_i}\oplus 0)S^{-1}$, where $x_i=\operatorname{rk}(P_i)$. Let us write Q_i as

$$Q_i=S_i\begin{bmatrix} A_i & B_i \\ C_i & D_i\end{bmatrix}S_i^{-1},\qquad A_i\in\mathbb{C}^{x_i\times x_i},\quad D_i\in\mathbb{C}^{(m_i-x_i)\times(m_i-x_i)}.$$

This proves that P_i and Q_i can be written as in (3). Next, we will prove that relations (4)–(7) hold. Using the last equality of (2) and having in mind that P_i and Q_i are idempotent we get

$$ab(P_i+Q_i-P_iQ_i-Q_iP_i)=\mu_i\nu_i I_{m_i}.$$

Using representations (3) we get

$$\begin{bmatrix} I_{x_i} & 0 \\ 0 & 0\end{bmatrix}+\begin{bmatrix} A_i & B_i \\ C_i & D_i\end{bmatrix}-\begin{bmatrix} A_i & B_i \\ 0 & 0\end{bmatrix}-\begin{bmatrix} A_i & 0 \\ C_i & 0\end{bmatrix}=\frac{\mu_i\nu_i}{ab}\begin{bmatrix} I_{x_i} & 0 \\ 0 & I_{m_i-x_i}\end{bmatrix}.$$

The upper-left and lower-right blocks prove (4) and (7). Now we use $Q_i^2=Q_i$. If we define $\rho_i=(\mu_i\nu_i)/(ab)$, from

$$\begin{bmatrix} (1-\rho_i)I_{x_i} & B_i \\ C_i & \rho_i I_{m_i-x_i}\end{bmatrix}\begin{bmatrix} (1-\rho_i)I_{x_i} & B_i \\ C_i & \rho_i I_{m_i-x_i}\end{bmatrix}=\begin{bmatrix} (1-\rho_i)I_{x_i} & B_i \\ C_i & \rho_i I_{m_i-x_i}\end{bmatrix}$$

we get

$$(1-\rho_i)^2 I_{x_i}+B_iC_i=(1-\rho_i)I_{x_i},\qquad C_iB_i+\rho_i^2 I_{m_i-x_i}=\rho_i I_{m_i-x_i}.$$

This proves (5) and (6). □

Let us observe that in (1), blocks P_0 and Q_0 may be absent.

The following corollary is a simple consequence of former Lemma 1.

Corollary 1. *Let $P,Q\in\mathbb{C}^{n\times n}$ be two idempotents matrices such that $PQ\neq QP$ and let $a,b\in\mathbb{C}\setminus\{0\}$ such that $aP+bQ$ is diagonalizable. If $\lambda\in\sigma(aP+bQ)\setminus\{0,a,b,a+b\}$, then there exists $\mu\in\sigma(aP+bQ)$ satisfying $a+b=\lambda+\mu$.*

We shall need the following well known result, sometimes known as the polynomial spectral mapping theorem (see e.g., [15, Theorem 9.33]):

Theorem 2. *For every matrix A and every polynomial p, one has $\sigma(p(A))=p(\sigma(A))$.*

In next result we relation the spectrum of the difference of two idempotents with the spectrum of a linear combination of these idempotents, provided this linear combination is diagonalizable.

Theorem 3. *Let $P,Q\in\mathbb{C}^{n\times n}$ be two idempotent matrices such that $PQ\neq QP$ and let $a,b\in\mathbb{C}\setminus\{0\}$ such that $aP+bQ$ is diagonalizable.*

(i) *If $\mu\in\sigma(aP+bQ)\setminus\{0,a,b,a+b\}$, there exists $\lambda\in\sigma(P-Q)$ such that $\frac{\mu(a+b-\mu)}{ab}=\lambda^2$.*

(ii) *If $\lambda\in\sigma(P-Q)\setminus\{0,-1,1\}$, then the roots of the polynomial $x^2-(a+b)x+\lambda^2ab$ are eigenvalues of $aP+bQ$.*

Proof. Represent P and Q as in Lemma 1.

(i) Pick any $\mu \in \sigma(aP + bQ) \setminus \{0, a, b, a + b\}$. By Theorem 1 and expression (1), there exists $i \in \{1, \ldots, k\}$ such that $\sigma(aP_i + bQ_i) = \{\mu, a + b - \mu\}$. Applying the last relation of (2) we have

$$\frac{\mu(a + b - \mu)}{ab} \in \sigma[(P - Q)^2].$$

The polynomial spectral mapping theorem finishes the proof of this item.

(ii) Pick any $\lambda \in \sigma(P - Q) \setminus \{0, -1, 1\}$. Since $\lambda \notin \{0, -1, 1\}$, by Theorem 1 we have that $\lambda \notin \sigma(P_0 - Q_0)$, hence there exists $i \in \{1, \ldots, k\}$ with $\lambda \in \sigma(P_i - Q_i)$. By the polynomial spectral mapping theorem we have $\lambda^2 \in \sigma[(P - Q)^2]$. By (2), there exist $\mu, \nu \in \sigma(aP_i + bQ_i)$ such that $a + b = \mu + \nu$ and $\lambda^2 = (\mu\nu)/(ab)$, and therefore, μ and ν are the roots of the polynomial $x^2 - (a + b)x + \lambda^2 ab$. The proof finishes by recalling $\sigma(aP_i + bQ_i) \subset \sigma(aP + bQ)$. □

Theorem 4. *Let $P, Q \in \mathbb{C}^{n\times n}$ be two idempotent matrices such that $PQ \neq QP$ and let $a, b, a', b' \in \mathbb{C} \setminus \{0\}$ such that $aP + bQ$ and $a'P + b'Q$ are diagonalizable. If $\mu \in \sigma(aP + bQ) \setminus \{0, a, b, a + b\}$, then the roots of the polynomial $x^2 - (a' + b')x + \frac{\mu(a+b-\mu)}{ab} a'b'$ are eigenvalues of $a'P + b'Q$.*

Proof. Let $\mu \in \sigma(aP + bQ) \setminus \{0, a, b, a + b\}$. By item (i) of Theorem 3 there exists $\lambda \in \sigma(P - Q)$ such that $\frac{\mu(a+b-\mu)}{ab} = \lambda^2$. Observe that $\lambda \notin \{0, -1, 1\}$ since $\mu \notin \{0, a, b, a + b\}$. By item (ii) of Theorem 3, the roots of the polynomial $x^2 - (a' + b')x + \lambda^2 a'b'$ are eigenvalues of $a'P + b'Q$. □

Next theorem concerns with the spectrum of PQ when P and Q satisfy Lemma 1.

Theorem 5. *Let $P, Q \in \mathbb{C}^{n\times n}$ be two idempotent matrices such that $PQ \neq QP$ and let $a, b \in \mathbb{C} \setminus \{0\}$ such that $aP + bQ$ is diagonalizable.*

(i) *If $\lambda \in \sigma(PQ) \setminus \{0, 1\}$, then the roots of the polynomial $x^2 - (a+b)x + ab(1-\lambda)$ are eigenvalues of $aP + bQ$.*

(ii) *If $\mu \in \sigma(aP + bQ) \setminus \{0, a, b, a + b\}$, then $1 - [\mu(a + b - \mu)]/(ab) \in \sigma(PQ)$.*

Proof. Let us represent P and Q as in (1). By (3) and (4) we have

$$PQ = S\left(\oplus_{i=1}^{k} P_iQ_i \oplus P_0Q_0\right)S^{-1}, \qquad P_iQ_i = S_i \begin{bmatrix} (1 - \rho_i)I_{x_i} & B_i \\ 0 & 0 \end{bmatrix} S^{-1} \tag{19}$$

for all $i = 1, \ldots, k$, with $\rho_i = (\mu_i\nu_i)/(ab)$. On the other hand, since P_0 and Q_0 are two commuting idempotents, P_0Q_0 is another idempotent, and therefore, $\sigma(P_0Q_0) \subset \{0, 1\}$.

(i) Pick any $\lambda \in \sigma(PQ) \setminus \{0, 1\}$. From (19), there exists $i \in \{1, \ldots, k\}$ such that $\lambda = 1 - \rho_i$. By item (ii) of Lemma 1, there exists two eigenvalues of $aP + bQ$, say μ and ν, such that $\rho_i = (\mu\nu)/(ab)$ and $\mu + \nu = a + b$. Hence

$$\mu + \nu = a + b, \qquad \text{and} \qquad \mu\nu = ab(1 - \lambda).$$

Therefore, μ and ν are the roots of the polynomial $x^2 - (a + b)x + ab(1 - \lambda)$.

(ii) Pick any $\mu \in \sigma(aP + bQ) \setminus \{0, a, b, a + b\}$. From Lemma 1 and Theorem 1, there exists $i \in \{1, \ldots, k\}$ such that μ and $a + b - \mu$ are eigenvalues of $aP_i + bQ_i$. Hence (19) implies $1 - [\mu(a + b - \mu)]/(ab) \in \sigma(P_iQ_i) \subset \sigma(PQ)$. □

Koliha and Rakočević proved in [12] that for p, q two nontrivial projections in a C^*-algebra (a projection f in a C^*-algebra satisfies $f^2 = f = f^*$) and $\lambda \in \mathbb{C} \setminus \{0, 1, -1\}$, then $\lambda \in \sigma(p - q)$ if and only if $1 - \lambda^2 \in \sigma(pq)$. Observe that we obtain the same relations when in Theorem 5 we substitute $a = 1$, $b = -1$. See [7] for a further generalization of the result of Koliha and Rakočević.

Theorem 6. *Let $P, Q \in \mathbb{C}^{n\times n}$ be two idempotent matrices such that $PQ \neq QP$ and let $a, b \in \mathbb{C} \setminus \{0\}$ such that $aP + bQ$ is diagonalizable.*

(i) *If* $\lambda \in \sigma(PQP) \setminus \{0,1\}$*, then the roots of the polynomial* $x^2 - (a+b)x + ab(1-\lambda)$ *are eigenvalues of* $aP + bQ$*.*

(ii) *If* $\lambda \in \sigma(aP+bQ) \setminus \{0,a,b,a+b\}$*, then* $1 - [\mu(a+b-\mu)]/(ab) \in \sigma(PQP)$*.*

Proof. Let us represent P and Q as in (1). By (3) and (4) we have

$$PQP = S\left(\oplus_{i=1}^{k} P_iQ_iP_i \oplus P_0Q_0\right)S^{-1}, \qquad P_iQ_iP_i = S_i \begin{bmatrix} (1-\rho_i)I_{x_i} & 0 \\ 0 & 0 \end{bmatrix} S_i^{-1},$$

for $i = 1, \ldots, k$ with $\rho_i = (\mu_i\nu_i)/(ab)$. The proof finishes as in the proof of Theorem 5. □

Theorem 7. *Let* $P, Q \in \mathbb{C}^{n\times n}$ *be two idempotents matrices such that* $PQ \neq QP$ *and let* $a, b \in \mathbb{C} \setminus \{0\}$ *such that* $aP + bQ$ *is diagonalizable.*

(i) *Let* $\lambda \in \sigma(PQ - QP) \setminus \{0\}$*. There exist* $\mu, \nu \in \sigma(aP + bQ)$ *such that*

$$\lambda^2 = -\frac{\mu\nu}{ab}\left(1 - \frac{\mu\nu}{ab}\right) \qquad \text{and} \qquad \mu + \nu = a + b. \tag{20}$$

(ii) *If* $\mu \in \sigma(aP+bQ) \setminus \{0,a,b,a+b\}$*, then there exists* $\lambda \in \sigma(PQ - QP)$ *such that*

$$-\frac{\mu(a+b-\mu)}{ab}\left(1 - \frac{\mu(a+b-\mu)}{ab}\right) = \lambda^2.$$

Proof. Let us represent P and Q as in (1). We have that

$$PQ - QP = S\left(\oplus_{i=1}^{k}(P_iQ_i - Q_iP_i) \oplus 0\right)S^{-1}.$$

By (3) and (4), for any $i = 1, \ldots, k$, one has

$$P_iQ_i - Q_iP_i = S_i\left(\begin{bmatrix} (1-\rho_i)I_{x_i} & B_i \\ 0 & 0 \end{bmatrix} - \begin{bmatrix} (1-\rho_i)I_{x_i} & 0 \\ C_i & 0 \end{bmatrix}\right)S_i^{-1} = S_i \begin{bmatrix} 0 & B_i \\ -C_i & 0 \end{bmatrix} S_i^{-1},$$

where $\rho_i = (\mu_i\nu_i)/(ab)$. Now,

$$(P_iQ_i - Q_iP_i)^2 = S_i \begin{bmatrix} 0 & B_i \\ -C_i & 0 \end{bmatrix}\begin{bmatrix} 0 & B_i \\ -C_i & 0 \end{bmatrix} S_i^{-1} = S_i \begin{bmatrix} -B_iC_i & 0 \\ 0 & -C_iB_i \end{bmatrix} S_i^{-1}.$$

And (5), (6) lead to

$$(P_iQ_i - Q_iP_i)^2 = -\rho_i(1-\rho_i)I_{m_i}. \tag{21}$$

(i) Pick any $\lambda \in \sigma(PQ - QP) \setminus \{0\}$. We have $\lambda^2 \in \sigma[(PQ-QP)^2] \setminus \{0\}$, and thus, there exists $i \in \{1, \ldots, k\}$ such that $\lambda^2 = -\rho_i(1-\rho_i)$. So, there exist $\mu, \nu \in \sigma(aP+bQ)$ satisfying (20).

(ii) Pick any $\mu \in \sigma(aP+bQ) \setminus \{0,a,b,a+b\}$. There exists $i \in \{1,\ldots,k\}$ such that $\mu \in \sigma(aP_i + bQ_i)$. If we define $\rho = \mu(a+b-\mu)/(ab)$, then (21) and Lemma 1 ensure that $-\rho(1-\rho) \in \sigma[(PQ-QP)^2]$. The polynomial spectral mapping theorem finishes the proof. □

Let us show how the previous results fit in the literature concerning on linear combinations of two idempotents.

In [2] it was characterized when $aP + bQ$ is idempotent provided P and Q are idempotents and $a, b \in \mathbb{C} \setminus \{0\}$. We will show how to derive this result (if $PQ \neq QP$) by using Lemma 1. Since $aP + bQ$ is idempotent, $\sigma(aP+bQ) \subset \{0,1\}$. Since $PQ \neq QP$, then $\sigma(aP+bQ)$ is not a singleton. From Lemma 1, we can write $P = S(P_1 \oplus P_0)S^{-1}$, $Q = S(Q_1 \oplus Q_0)S^{-1}$ with $P_0Q_0 = Q_0P_0$, $P_1Q_1 \neq Q_1P_1$, $a + b = 1$ and $(P_1 - Q_1)^2 = 0$. If the blocks P_0, Q_0 were present, then by a simultaneous diagonalization, we would get that $a = b = 1$, or $a = 1, b = -1$, or $a = -1, b = 1$ which would contradict $a + b = 1$. Hence $(P - Q)^2 = 0$.

In [1] the author characterized when $c_1A_1 + c_2A_2 + c_3A_3$ is idempotent provided A_1, A_2, A_3 are idempotents, $A_2A_3 = A_3A_2 = 0$, and $c_1, c_2, c_3 \in \mathbb{C} \setminus \{0\}$ (this paper generalized to [4]). We will not give the whole solution, but only deduce a consequence from Lemma 1. Let $P = A_1$, $Q = c_1A_1 + c_2A_2 + c_3A_3$, $a = -c_1$, and $b = 1$. Now, $aP + bQ = c_2A_2 + c_3A_3$ is diagonalizable and $\sigma(aP + bQ) = \{c_2, c_3\}$ because A_2, A_3 are nonzero idempotents with $A_2A_3 = A_3A_2 = 0$. If $PQ \neq QP$, and $c_2 \neq c_3$, then Lemma 1 yields $a + b = c_2 + c_3$, i.e., $1 = c_1 + c_2 + c_3$.

If P and Q are idempotents and $a, b \in \mathbb{C} \setminus \{0\}$, then the authors of [5] studied when $(aP + bQ)^{k+1} = aP + bQ$ for a fixed $k \in \mathbb{N}$. In [5] it was proved that a matrix X satisfies $X^{k+1} = X$ if and only if X is diagonalizable and $\sigma(X) \subset \{0\} \cup \sqrt[k]{1}$. Therefore, Lemma 1 implies that if $PQ \neq QP$ there exists $\alpha, \beta \in \{0\} \cup \sqrt[k]{1}$ such that $\alpha + \beta = a + b$ and $\alpha \neq \beta$.

Also, there has been many results concerning the invertibility of expressions involving two idempotents [3, 9, 11, 14]. As an example, we shall prove the following result: If $P, Q \in \mathbb{C}^{n\times n}$ are two idempotents such that $P + Q$ is diagonalizable and $P - Q$ is nonsingular, then $P + Q$ and $I_n - PQ$ are nonsingular [3, 11]. Let us write P and Q as in (1). By a similar argument as in Theorem 1, there exists a nonsingular $S \in \mathbb{C}^{m_0\times m_0}$ such that $P_0 = S\text{diag}(\lambda_1, \ldots, \lambda_{m_0})S^{-1}$ and $Q_0 = S\text{diag}(\mu_1, \ldots, \mu_{m_0})S^{-1}$, where $\lambda_i, \mu_j \in \{0, 1\}$ for $i, j \in \{1, \ldots, m_0\}$. As $P-Q$ is nonsingular, then $\lambda_i \neq \mu_i$ for all $i \in \{1, \ldots, m_0\}$, and therefore $\lambda_i + \mu_i = 1 - \lambda_i\mu_i = 1$ for all $i \in \{1, \ldots, m_0\}$, which implies the nonsingularity of $P_0 + Q_0$ and $I_{m_0} - P_0Q_0$. If $P + Q$ were singular, by (1), there would exist $i \in \{1, \ldots, k\}$ such that $P_i + Q_i$ is singular, by the second expression of (2), one has $\mu_i = 0$ or $\nu_i = 0$, and by the last expression of (2), one gets $(P_i - Q_i)^2 = 0$, hence $P_i - Q_i$ is singular, in contradiction with the nonsingularity of $P - Q$. If $I_n - PQ$ were singular, by (1), there would exist $j \in \{1, \ldots, k\}$ such that $I_{m_j} - P_jQ_j$ is singular. By (3) and (4)

$$I_{m_j} - P_jQ_j = S_j \begin{bmatrix} \frac{\mu_j\nu_j}{ab} I_{x_j} & -B_j \\ 0 & I_{m_j - x_j} \end{bmatrix} S_j^{-1},$$

which in view of the singularity of $I_{m_j} - P_jQ_j$ we get $\mu_j = 0$ or $\nu_j = 0$, and as before, we arrive at a contradiction.

Example: Let

$$P = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \quad \text{and} \quad Q = \frac{1}{2}\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}.$$

These matrices satisfy $P^2 = P$, $Q^2 = Q$, and $PQ \neq QP$. Furthermore, if $a, b \in \mathbb{R} \setminus \{0\}$, then $aP + bQ$ is diagonalizable (because $aP + bQ$ is Hermitian). One trivially has $\sigma(P - Q) = \{1/\sqrt{2}, -1/\sqrt{2}\}$, and by Theorem 3, we obtain that the roots of the polynomial $x^2-(a+b)x+ab/2$ are the eigenvalues of $aP + bQ$, which can be verified by simplifying $\det(aP + bQ - \lambda I_2)$. On the other hand, we can quickly compute than $\sigma(PQ) = \sigma(PQP) = \{0, 1/2\}$, and Theorem 5 or Theorem 6 lead again that the roots of $x^2 - (a + b)x + ab/2$ are the eigenvalues of $aP + bQ$. Finally, a trivial computation shows $\sigma(PQ - QP) = \{\mathrm{i}/2, -\mathrm{i}/2\}$, and Theorem 7 yields that there exists $\mu, \nu \in \sigma(aP + bQ)$ such that $-\frac{1}{4} = -\frac{\mu\nu}{ab}\left(1 - \frac{\mu\nu}{ab}\right)$ and $\mu + \nu = a + b$, which reduce to $\mu\nu = ab/2$ and $\mu + \nu = a + b$, in other words, μ, ν are the roots of $x^2 - (a + b)x + ab/2$.

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