Advanced aspects in multi-criteria decision-making: coping with uncertainty using Probability Theory

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Decision-making challenges in a changing world: an international overview on academic and professional environments

UNIVERSITÀ DEGLI STUDI DI PALERMO

Part I
An Introduction to Probability Theory

Random Variables

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2. Toss a coin until a "head" appears. How many tosses?
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2. Toss a coin until a "head" appears. How many tosses?
   If X denotes the number of heads, then $X = 0, 1, \ldots$
3. Pick randomly an apple. Which is its weight?
   If X denotes the weight, then $0 < X < \infty$.

Classification of random variables

Discrete

- Toss a coin 20 times. How many heads?
- Toss a coin until a head appears. How many tosses?
- How many visitors has a web page in a day?
- ...

The value of the random variable "jumps".

Continuous

- The weight of an apple.
- The time of a flight travel.
- The height of a citizen of a given country.
- ...

The value of the random variable "flows".

Discrete random variables

Let X be a discrete random variable with values $x_1, x_2, \ldots$

X is determined by giving $\text{pr}(X = x_1), \text{pr}(X = x_2), \ldots$

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- $X = 0$. No heads $\rightarrow$ Tail-Tail. $\text{pr}(X = 0) = (1 - p)^2$.
- $X = 1$. One head $\rightarrow$ Tail-Head or Head-Tail. $\text{pr}(X = 1) = 2p(1 - p)$.
- $X = 2$. Two heads $\rightarrow$ Head-Head. $\text{pr}(X = 2) = p^2$.

Discrete distributions

Example 1

Toss a coin n times.
$p = \text{pr}(\text{Head})$.
If X is the number of heads, then

$$\text{pr}(X = k) = ?$$

Example 2

There are n persons in a room.
Let p be the proportion of left-handed.
If X is the number of left-handed, then

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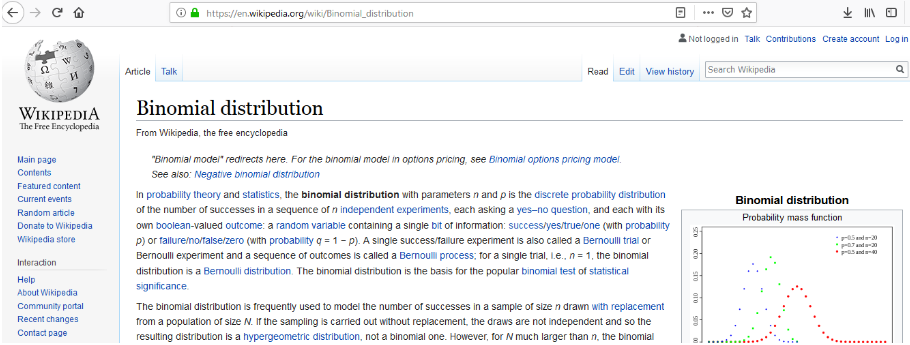
These problems are exactly the same!

What is a probability distribution?

- There are few different models.
- Useful for a wide range of problems.
- No necessity of deducing formulas.
- For example, the two previous problems can be modelled by the "binomial distribution".



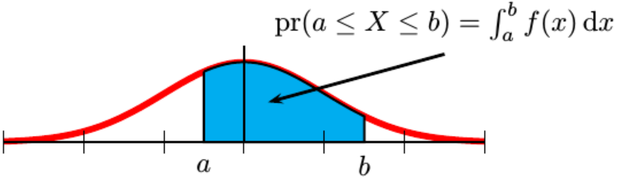
Continuous random variables

Let X be a continuous random variable.
X is determined by fixing $f : \mathbb{R} \to \mathbb{R}$ such that

$$\mathrm{pr}(a \leq X \leq b) = \int_a^b f(x)\,\mathrm{d}x.$$

This function f is said to be the **density function** of X.

In most situations, no necessity of computing any integral, since there are few models, and the integrals are very easy or are tabulated.



Expectation

Examples

- If we toss a fair coin many times, then the proportion of heads tends to $1/2$.
- If I weight many apples, and I compute the mean, then this mean tends to a concrete value.

These examples show the **law of large numbers**.

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These examples show the **law of large numbers**.

Let X be a random variable. We perform the random experiment many times obtaining $X_1, \dots, X_n$ and let $\overline{X}_n = (X_1 + \cdots + X_n)/n$. Intuitively, $\overline{X}_n$ will tend to a number. **Which? How can it be computed?**

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Definition of the **expectation**

Let X be a random variable. The expectation of X is defined as

- If X is discrete, $\mathbb{E}(X) = \sum_i x_i \,\mathrm{pr}(X = x_i)$.
- If X is continuous, $\mathbb{E}(X) = \int_{-\infty}^{\infty} x f(x)\,\mathrm{d}x$.

Law of large numbers

If $X_1, X_2, X_3, \ldots$ are independent random variables with the same distribution and μ is the common expectation, then for all $\varepsilon > 0$, one has

$$\lim_{n\to\infty} \operatorname{pr}\left(\left|\frac{X_1 + \cdots + X_n}{n} - \mu\right| < \varepsilon\right) = 1.$$

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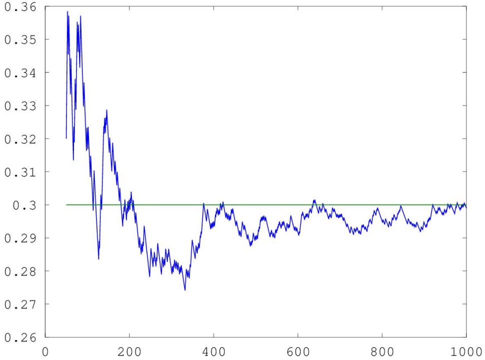

$p = 0.3.$
$\text{pr}(X_i = 1) = p.$
$\text{pr}(X_i = 0) = 1 - p.$
$\mathbb{E}(X_i) = p.$
$(X_1 + \cdots + X_n)/n \simeq \mathbb{E}(X_i).$

Variance

The variance of a random variable measures the degree of variability, the uncertainty, or the unpredictability.

Definition of the variance

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$$\mathbb{V}\mathrm{ar}(X) = \mathbb{E}\left[(X - \mathbb{E}(X))^2\right].$$

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Let X_1 be a continuous uniform random variable defined in $[0, 1]$.

Let X_2 be a continuous uniform random variable defined in $[0, 2]$.

Which random variable has a larger variance?

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Which random variable has a larger variance? X_2.

Properties of the expectation and variance

Let X and Y be two random variables and $k \in \mathbb{R}$.

Expectation

- $\mathbb{E}(X + Y) = \mathbb{E}(X) + \mathbb{E}(Y)$.
- $\mathbb{E}(kX) = k\mathbb{E}(X)$.

Variance

- If X and Y are independent, $\mathbb{V}\mathrm{ar}(X + Y) = \mathbb{V}\mathrm{ar}(X) + \mathbb{V}\mathrm{ar}(Y)$.
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Example – Mean

Let $X_1, \dots, X_n$ independent random variables with $\mathbb{E}(X_i) = \mu$ and $\mathbb{V}\mathrm{ar}(X_i) = \sigma^2$.

$$\mathbb{E}\left(\frac{X_1 + \cdots + X_n}{n}\right) = \mu, \quad \mathbb{V}\mathrm{ar}\left(\frac{X_1 + \cdots + X_n}{n}\right) = \frac{\sigma^2}{n}$$

The meaning of the variance

Chebyshev's inequality

If X is an **arbitrary** random variable, then

$$\mathrm{pr}\left(|X - \mathbb{E}(X)| \geq c\right) \leq \frac{\mathbb{V}\mathrm{ar}(X)}{c^2}.$$

The smaller $\mathbb{V}\mathrm{ar}(X)$ is, the more concentrated is X respect its expectation.

Чебышёв

The normal distribution

Definition of the **normal distribution**

It is a continuous distribution whose density function is

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}} \exp\left(-\frac{(x-\mu)^2}{2\sigma^2}\right).$$

The expectation is μ and the variance is σ^2.

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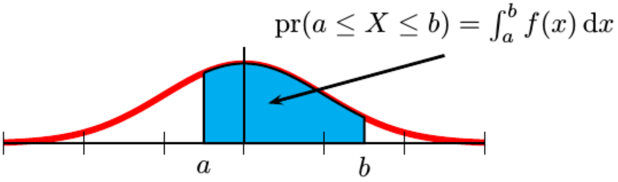
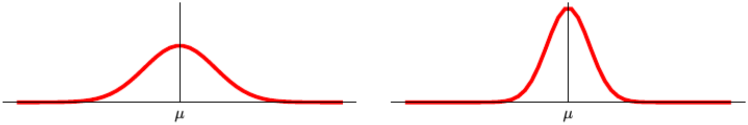


The definite integral of f is **always** computed by means of tables or a software.

Properties of the normal distribution

- The normal distribution is symmetric respect its expectation.
- It is very concentrated around the expectation.
- For large numbers, the probability is negligible.



The variance of the left distribution is larger than the variance of the right distribution.

Usefulness and meaning of the normal distribution

The normal distribution is ubiquitous. It appears in

- The height of adult male (and female) humans.
- The intelligent quotient.
- The weight of a concrete variety of apples.
- The time of a flight between two airports.
- The temperature at 1st May in a given city.
- The errors of measurements (Gauss).
- ...

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In fact, normal distribution approximates very well. Remember that its density function is defined for all $x \in \mathbb{R}$, but for an interval I far from the expectation, then $\mathrm{pr}(X \in I) \simeq 0$.

Meaning of the normal distribution. The central limit theorem

Under certain (fairly common) conditions, the mean of many random variables will have an approximately normal distribution.

Part II
An Introduction to AHP

What is the Analyty Hierarchy Process?

Decision making is becoming increasingly complex due to the large number of alternatives and multiple conflicting goals.

The Analytic Hierarchy Process (AHP) has been accepted as a leading multiattribute decision-aiding model.

Decisions arise in every type of application including

- Planning
- Economy
- Resource allocation
- Medical management
- Emergency situations
- ...

Pairwise comparisons

I want to submit a manuscript to one of the following two journals:

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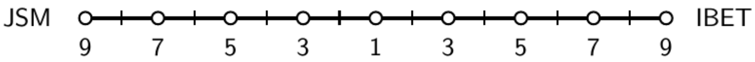

1 Equal importance.

3 Moderate importance of one over another.

5 Strong importance.

7 Very strong importance.

9 Extreme importance.

Reciprocal matrices

- Journal of Superb Mathematics
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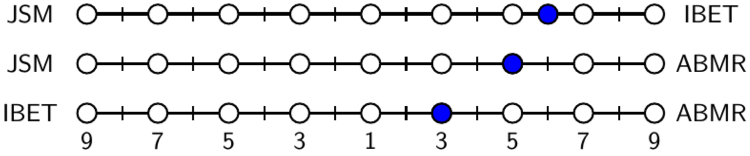

$$\begin{array}{c} \\ \text{JSM} \\ \text{IBET} \\ \text{ABMR} \end{array} \begin{array}{c} \begin{array}{ccc} \text{JSM} & \text{IBET} & \text{ABMR} \end{array} \\ \begin{bmatrix} 1 & 1/6 & 1/5 \\ 6 & 1 & 1/3 \\ 5 & 3 & 1 \end{bmatrix} \end{array}.$$

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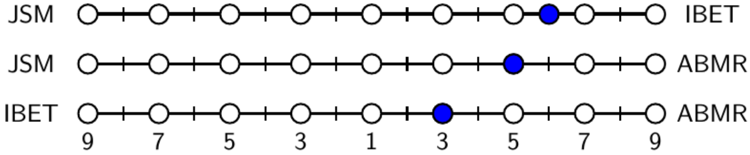

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Definition of reciprocal matrices

A square matrix $(a_{ij})_{i,j=1}^{n}$ is **reciprocal** when $a_{ij} > 0$ and $a_{ij} = 1/a_{ji}$ for all $i, j \in \{1, \ldots, n\}$.

Consistent matrices

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- I prefer ABMR 3 times over IBET.
- I prefer IBET 6 times over JSM.
- I prefer ABMR 5 times over JSM.

This is not "rational". I had to prefer ABMR 3×6 times over JSM!

$$3 \times 6 \neq 5.$$

Definition of consistent matrices

A matrix $A = (a_{ij})_{i,j=1}^{n} \in \mathbb{R}_{n,n}$ is consistent when $a_{ij} > 0$ and $a_{ij}a_{jk} = a_{ik}$ for all $i, j, k \in \{1, \ldots, n\}$.

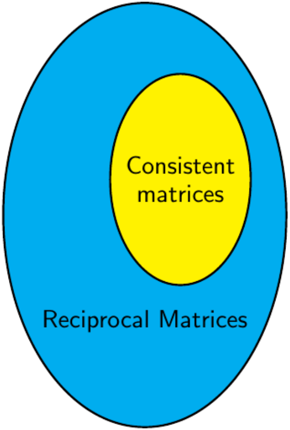
Reciprocal and consistent matrices

Any consistent matrix is a reciprocal matrix.

However, if A is a reciprocal matrix, it may happen that A is not consistent.

$$A = \begin{bmatrix} 1 & a & b \\ 1/a & 1 & c \\ 1/b & 1/c & 1 \end{bmatrix}$$

is reciprocal, but if $ac \neq b$, then A is not consistent.



The priority vector. Example

We start from the preferences (it's not usual).
Imagine that

- 50% of the days, I prefer tea.
- 40% of the days, I prefer milk.
- 10% of the days, I prefer coffee.

The normalised priority vector is $(0.5, 0.4, 0.1)$.
I prefer milk 4 times than coffee, tea 5 times than coffee, tea 5/4 times than milk.

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The entry (i, j) of A is v_i/v_j, being (v_1, v_2, v_3) the priority vector.
In particular, any column is a multiple of the priority vector.

The priority vector (consistent matrices)

Theorem

A matrix $A = (a_{ij}) \in \mathbb{R}_{n,n}$ *is consistent if and only if exists* $\mathbf{v} = (v_1, \cdots, v_n) \in \mathbb{R}^n$ *such that* $v_i > 0$ *and* $a_{ij} = v_i / v_j$ *for all* i, j.

Definition of the **priority vector**

This (normalised) vector $\mathbf{v}$ is important: It tells us how to rank the alternatives. It is called the priority vector.

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Theorem

If $A \in \mathbb{R}_{n,n}$ is consistent and $\mathbf{v}$ is a priority vector, then $A\mathbf{v} = n\mathbf{v}$.

Definition of an eigenvalue and eigenvector

Given a square matrix A, we say that $\lambda \in \mathbb{C}$ is an eigenvalue of A if exists a nonzero vector $\mathbf{v}$ such that $A\mathbf{v} = \lambda\mathbf{v}$. The vector $\mathbf{v}$ is called an eigenvector.

Inconsistency of reciprocal matrices

T. Saaty proposed a measure of the inconstency of a reciprocal matrix.

Theorem (Saaty, 1970)

Let $A \in \mathbb{R}_{n,n}$ be reciprocal. Let

$$\rho(A) := \max\{|\lambda| : \lambda \text{ is an eigenvalue of } A\}.$$

The following statements hold:

- $\rho(A) \leq n$.
- $\rho(A) = n$ *if and only if* A *is consistent.*

Saaty proposed to define that the inconsistency of a reciprocal matrix is acceptable if $n - \rho(A)$ is "small" in a precise sense that it is omitted here.

The priority vector (reciprocal matrices)

Let $A \in \mathbb{R}_{n,n}$ be reciprocal and assume that its consistency is acceptable.

The *Perron theorem* (1907) asserts that a real square matrix with positive entries has a unique largest real eigenvalue and that the corresponding eigenvector (unique up to a scalar multiple) can be chosen to have strictly positive components.

Definition of the priority vector

We define the priority vector of a reciprocal matrix as the eigenvector given by the Perron theorem.

A flowchart in AHP theory

First step: The expert build a reciprocal matrix A.

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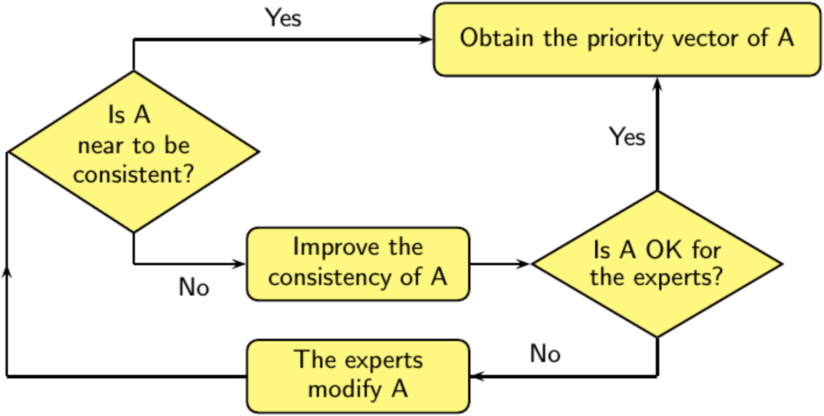

Modifying a reciprocal matrix to improve its consistency

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When the distance between these matrices is small.
We need a distance in the set of reciprocal matrices! A classical measure is the following:

Definition of the Frobenius norm

If $A = (a_{ij})$ and $B = (b_{ij})$ are matrices, then

$$\|A - B\|_F = \sum_{i,j} (a_{ij} - b_{ij})^2.$$

The Frobenius norm is not suitable in AHP

Example

$$A_1 = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix},\ B_1 = \begin{bmatrix} 1 & 2 \\ 1/2 & 1 \end{bmatrix},\ A_2 = \begin{bmatrix} 1 & 8 \\ 1/8 & 1 \end{bmatrix},\ A_2 = \begin{bmatrix} 1 & 9 \\ 1/9 & 1 \end{bmatrix}$$

- Matrix A_1 reflects the two criteria are equivalent, while B_1 reflects that the 1st criterion is twice more important than the 2nd criterion.
- The importance of the criteria in A_2 and B_2 are very close.
- Thus, in an intuitive point of view, the distance between A_1 and B_1 must be much greater than the distance between A_2 and B_2.

$$\|A_1 - B_1\|_F \simeq 1.118, \qquad \|A_2 - B_2\|_F \simeq 1.001.$$

Another distance

We define the following distance in the set of $n \times n$ reciprocal matrices:

Definition

$$d(A, B) = \|L(A) - L(B)\|_F,$$

where $L(A)$ is a $n \times n$ matrix whose (i, j) entry is $\log(a_{ij})$.

Example

$$A_1 = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}, \; B_1 = \begin{bmatrix} 1 & 2 \\ 1/2 & 1 \end{bmatrix}, \; A_2 = \begin{bmatrix} 1 & 8 \\ 1/8 & 1 \end{bmatrix}, \; A_2 = \begin{bmatrix} 1 & 9 \\ 1/9 & 1 \end{bmatrix}$$

$$d(A_1, B_1) \simeq 0.9803, \qquad d(A_2, B_2) \simeq 0.1666.$$

Also we need the mapping $E : M_{n,n} \to M_{n,n}$ given by $(E(A))_{ij} = \exp(a_{ij})$.

Linearisation process

1. The expert builds a reciprocal $n \times n$ matrix A.
2. We compute $B = L(A)$, i.e., $b_{ij} = \log(a_{ij})$.
3. Compute
$$M = \frac{1}{n}\left[(BU_n) - (BU_n)^T\right],$$
where "T" means the transposition and U_n is the $n \times n$ matrix all of whose components are 1.
4. Compute $X_A = E(M)$, i.e., $x_{ij} = \exp(m_{ij})$.

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$$M = \frac{1}{n}\left[(BU_n) - (BU_n)^T\right],$$
where "T" means the transposition and U_n is the $n \times n$ matrix all of whose components are 1.
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Linearisation process

1. The expert builds a reciprocal $n \times n$ matrix A.
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This process is based on the next result

Theorem

If A is reciprocal and X_A is computed as before, then X_A is consistent and $\mathrm{d}(A, X_A) = \min\{\mathrm{d}(A, C) : C \text{ is consistent}\}$.

Caution! It's possible that this matrix X_A doesn't reflect the requirements of the expert. In this, case, a feedback process must be performed (modifying X_A).

Linearisation process (bis)

1. The expert builds a reciprocal $n \times n$ matrix A.
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4. Compute $X_A = E(M)$, i.e., $x_{ij} = \exp(m_{ij})$.
5. A priority vector of X_A is $\mathbf{v} = (v_1, \ldots, v_n)$, where
$$v_i = \sqrt[n]{a_{i1} \cdots a_{in}}.$$

Part III
AHP and Probability Theory

Probability judgements

I want to buy a car and I can choose from A, B, C.

I prefer A twice B $\rightarrow \begin{bmatrix} 1 & 2 & \\ 1/2 & 1 & \\ & & 1 \end{bmatrix}$. The value of 2 is exact.

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$$\begin{bmatrix} 1 & a_{12} & \\ 1/a_{12} & 1 & \\ & & 1 \end{bmatrix}.$$

a_{12} is a random variable.

AHP and probalisitic related concepts

Remember that if X and Y are random variables and $k \in \mathbb{R}$, then

- $\mathbb{E}(X+Y) = \mathbb{E}(X) + \mathbb{E}(Y)$, $\mathbb{E}(kX) = k\mathbb{E}(X)$.
- $\mathbb{V}\mathrm{ar}(kX) = k^2\mathbb{V}\mathrm{ar}(X)$.
- If X and Y are independent, then $\mathbb{V}\mathrm{ar}(X+Y) = \mathbb{V}\mathrm{ar}(X) + \mathbb{V}\mathrm{ar}(Y)$.

But, in view of the properties of the reciprocal matrices and linearisation process

- If $A = (a_{ij})$ is reciprocal, then $a_{ji} = 1/a_{ij}$.
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Definition of the geometric expectation and variance

Given a positive random variable X,

- The geometric expectation of X is $\mathbb{G}(X) = \exp(\mathbb{E}(\log X))$.
- The geometric variance of X is $\mathbb{V}\mathrm{ar}_g(X) = \mathbb{V}\mathrm{ar}(\log X)$.

Properties and AHP

Let X, Y be random variables and $k \in \mathbb{R}$

- $\mathbb{G}(XY) = \mathbb{G}(X)\mathbb{G}(Y)$, $\mathbb{G}(X^k) = \mathbb{G}(X)^k$.
- $\mathbb{V}\mathrm{ar}_g(X^k) = k^2 \mathbb{V}\mathrm{ar}_g(X)$.
- If X and Y are independent, then $\mathbb{V}\mathrm{ar}_g(XY) = \mathbb{V}\mathrm{ar}_g(X)\mathbb{V}\mathrm{ar}_g(Y)$.

Which allows to study reciprocal matrices and the linearisation process:

- If $A = (a_{ij})$ is reciprocal, then $a_{ji} = 1/a_{ij}$.
- The ith coordinate of the priority vector of the nearest consistent matrix to A is $\sqrt[n]{a_{i1} \cdots a_{n1}}$.

An example

Example

a_{12} is a random variable such that $\text{pr}(a_{12} = 1) = \text{pr}(a_{12} = 2) = 1/2$.

$$\mathbb{E}(a_{12}) = \frac{1+2}{2} = 1.5, \quad \mathbb{E}(a_{21}) = \frac{1+1/2}{2} = 0.75,$$

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$$a_{21} = \frac{1}{a_{12}}, \qquad \mathbb{E}(a_{21}) \neq \frac{1}{\mathbb{E}(a_{12})}.$$

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On the other hand,

$$\mathbb{G}(a_{12}) = \sqrt{1 \cdot 2} = \sqrt{2}, \qquad \mathbb{G}(a_{21}) = \sqrt{1 \cdot \frac{1}{2}} = \frac{1}{\sqrt{2}} = \frac{1}{\mathbb{G}(a_{12})}.$$

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In the 2nd situation, the expert's doubts are much smaller than in the 1st situation.

However, $\mathbb{V}\text{ar}(a_{12}) = \mathbb{V}\text{ar}(b_{12})$ — it follows from $\mathbb{V}\text{ar}(X + k) = \mathbb{V}\text{ar}(X)$, where X is a random variable and k is a constant.

In contrast, one has $\mathbb{V}\text{ar}_g(a_{12}) = 0.12011$, $\mathbb{V}\text{ar}_g(b_{12}) = 0.00347$.

The log-normal distribution and AHP

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Example (numerical, Octave)

I have doubts about a_{12}. I want that a_{12} is around 2.

Assume that $X = \log(a_{12})$ follows a normal distribution with mean $\log(2)$.

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generates 10 numbers randomly, normally distributed with mean $\log(2)$ and variance 1.5^2.

We obtain $6.7, 0.14, 0.43, 2.3, 0.18, 12.9, 3.1, 1.2, 9.4, 1.5$ Too much variance!

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We can choose smaller variances until we accept the randomly numbers.

Many thanks

These slides are available at

`http://personales.upv.es/jbenitez/investigacion.html`