# Periodic distribution


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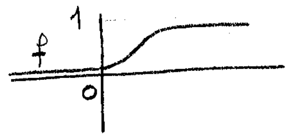
# I Some preliminary facts

A <u>distribution</u> is some kind of operation defined on functions. Those functions are usually a class of smooth enough ones, in a sense that will be described here. It is necessary, then, to start by looking at some differentiable ($\equiv$ smooth) functions which will serve as « test » functions.

We shall consider functions defined on $\mathbb{R}$ (with real or complex values). Any such function is said to be <u>infinitely differentiable</u> whenever it has derivatives of all orders, which shall be denoted by $f^{(0)} \equiv f,\ f^{(1)},\ f^{(2)},\ \dots$, or $D^0 f,\ D^1 f,\ D^2 f, \dots$

**I.1. <u>Proposition</u>** : there exists an infinitely differentiable function $f: \mathbb{R} \to \mathbb{R}$ such that $f(x) = 0,\ \forall x \leq 0,\ f(x) > 0,\ \forall x > 0.$

<u>Proof</u>: Choose
$$f(x) = \begin{cases} 0, & x \leq 0 \\ \exp(-1/x), & x > 0 \end{cases}$$

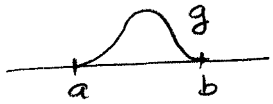



Note that for $x > 0$
$$\exp(1/x) = \sum_0^\infty \left(\frac{1}{x}\right)^n \cdot \frac{1}{n!} \geq \frac{1}{x^n \cdot n!}, \quad \forall n \geq 0$$
Then
$$\exp(-1/x) \leq x^n \cdot (n!), \quad \forall n \geq 0,\ \forall x > 0$$
It follows that $f$ is continuous at $x = 0$.
It is easy to prove, by induction, that
$$f^{(k)}(x) = P_{2k}(x^{-1}) \cdot f(x), \quad x > 0$$
where $P_{2k}$ is a polynomial of degree $2k$ in one variable (for example, $f'(x) = x^{-2} \cdot \exp(-x^{-1}),\ f''(x) = (-2x^{-3} + x^{-4})\exp(-x^{-1}), \dots$ )

Assume we already proved that $f^{(k)}(0)$ exists (and then obviously has value $0$). Then, for $x > 0$
$$\frac{f^{(k)}(x) - f^{(k)}(0)}{x} = \frac{f^{(k)}(x)}{x} = \frac{1}{x} \cdot P_{2k}(x^{-1}) \cdot f(x) \leq$$
$$\leq \frac{1}{x} \cdot P_{2k}(x^{-1}) \cdot x^{2k+2} \cdot (2k+2)!$$
and so $\exists\, f_+^{(k+1)}(0) = 0$. $\square$

I.2 Remark: $f$ has a peculiar property: it is infinitely differentiable and all derivatives at $0$ are $0$. Should $f$ be a power series in some neighbourhood of $0$, $f$ shall be $0$ in this neighbourhood, and this is not the case.
It follows that $f$, although infinitely differentiable, is not a power series (even is not a power series in an arbitrarily small neighbourhood of $0$)

I.3 Corollary Given $a<b$, there exists an infinitely differentiable function $g: \mathbb{R} \to \mathbb{R}$ such that



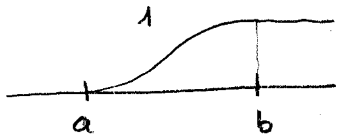

$$g(x) = 0, \quad x \notin ]a,b[$$
$$g(x) > 0, \quad x \in ]a,b[$$

$g$ sometimes is called a (infinitely differentiable) bump on $\mathbb{R}$, with support in $[a,b]$.

Proof: This is simple by using $f$ from Proposition I.1: define

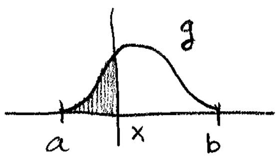
$$g(x) = f(x-a) \cdot f(b-x), \quad \forall x \in \mathbb{R}. \quad \square$$

I.4 Corollary Given $a<b$, there exists an infinitely differentiable function $h: \mathbb{R} \to \mathbb{R}$ such that




$$h(x) = 0, \quad x \leq a$$
$$0 < h(x) < 1, \quad x \in ]a,b[$$
$$h(x) = 1, \quad x \geq b$$

Proof: It is again straightforward just using $g$:




Define $h_1(x) = \int_{-\infty}^{x} g(t)\,dt$, $x \in \mathbb{R}$

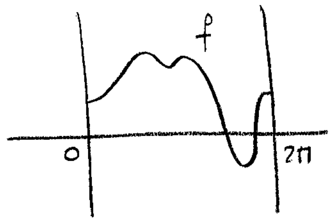
Now choose $c := \int_{-\infty}^{b} g(t)\,dt$

and define $h(x) := \frac{1}{c} h_1(x)$, $x \in \mathbb{R}$. $\square$

II **Some spaces of continuous functions**

II.1 A <u>periodic function</u> (of period $a>0$) is a real or complex function defined on $\mathbb{R}$ such that $f(x)=f(a+x)$, $\forall x\in\mathbb{R}$.
By an appropriate change of variable we can always assume that $a=2\pi$.
From now on a <u>periodic</u> function means a $2\pi$-periodic function.

$$\mathcal{C}\equiv\{f:\mathbb{R}\to\mathbb{C},\ \text{continuous},\ (2\pi)\text{-periodic}\}$$




<u>Remark</u>: A natural identification allows us to think an element in $\mathcal{C}$ as a continuous function defined on $T$ ($\equiv$ the unit circumference in the complexplane, i.e., the border of the unit disk).

II.2. A <u>norm</u> on $\mathcal{C}$ is defined as

$$\|f\|_\infty\equiv\sup_{x\in[0,2\pi]}|f(x)|=\sup_{x\in\mathbb{R}}|f(x)|=\max_{x\in\mathbb{R}}|f(x)|$$

It is obvious that $\|\cdot\|_\infty$ is in fact well defined, and it is a norm, in the sense of having

$$\|f\|_\infty\geqslant 0$$
$$\|f\|_\infty=0\iff f\equiv 0$$
$$\|\lambda f\|_\infty=|\lambda|\cdot\|f\|_\infty$$
$$\|f+g\|_\infty\leqslant\|f\|_\infty+\|g\|_\infty$$

for $\lambda\in\mathbb{C}$, $f,g\in\mathcal{C}$.

II.3 <u>Proposition</u> $(\mathcal{C},\|\cdot\|_\infty)$ is a complete normed space (i.e., a Banach space)

<u>Proof</u>: Let $(f_n)_1^\infty$ be a Cauchy sequence in $(\mathcal{C},\|\cdot\|_\infty)$.
This means $\quad\|f_n-f_m\|_\infty<\varepsilon$ for $n,m$ big enough ($\varepsilon$ was a positive a priori given number). In particular

$$|f_n(x)-f_m(x)|<\varepsilon,\ \forall n,m\ \text{big enough},\ x\in\mathbb{R}$$

By passing to the limit as $m\to\infty$, $|f_n(x)-f(x)|\leqslant\varepsilon$, $\forall n$ big enough, $x\in\mathbb{R}$.
It follows that $\|f_n-f\|_\infty\leqslant\varepsilon$, $\forall n$ big enough. A standard argument proves that $f\in\mathcal{C}$. $\square$

<u>Remark</u>: It is easy to identify here the argument about the continuity of the uniform limit of continuous functions. To be precise, we can

make the proof a little bit more detailed:

$(f_n)$ is a Cauchy sequence in $(\mathcal{C}, \|\cdot\|_\infty)$. Then, given $\varepsilon>0$, there exists $n_0 \in \mathbb{N}$ such that $\|f_n - f_m\|_\infty < \varepsilon$, $\forall n, m \geq n_0$. Observe that this means $|f_n(x) - f_m(x)| < \varepsilon$, $\forall n, m \geq n_0$, $\forall x \in \mathbb{R}$. It follows that $(f_n(x))_{n=1}^{\infty}$ is a Cauchy sequence in $\mathbb{C}$, hence convergent (to some $f(x)$). It is enough to let $n \to \infty$ to get $|f(x) - f_m(x)| \leq \varepsilon$, $\forall m \geq n_0$, $\forall x \in \mathbb{R}$. [*]

Observe, too, that $f$ is continuous:

Let $x_0 \in \mathbb{R}$. As $f_{n_0}$ is continuous, given $\varepsilon>0$ there exists $\delta>0$ such that $|x - x_0| < \delta \Rightarrow |f_{n_0}(x) - f_{n_0}(x_0)| < \varepsilon$. Then

$$|f(x) - f(x_0)| \leq |f(x) - f_{n_0}(x)| + |f_{n_0}(x) - f_{n_0}(x_0)| + |f_{n_0}(x_0) - f(x_0)| \leq$$
$$\leq \varepsilon + \varepsilon + \varepsilon = 3\varepsilon$$

This means that $f$ is continuous at $x_0$.

Moreover, from [*], $\|f - f_m\| \leq \varepsilon$, $\forall m \geq m_0$

So $(f_n) \longrightarrow f$ in $(\mathcal{C}, \|\cdot\|_\infty)$, as it is obvious that $f$ is also $2\pi$-periodic. □

## II.4. Linear functionals on a vector space

Let $X$ be a vector space (on the field of real or complex numbers; denote this field by $\mathbb{K}$). A function $T: X \to \mathbb{K}$ is <u>linear</u> whenever

$$T[x+y] = T[x] + T[y]$$
$$T[\lambda . x] = \lambda T[x],$$
$$\forall x, y \in X, \ \forall \lambda \in \mathbb{K}$$

<u>Remark</u>: Obviously, in such a case $T[0] = 0$, and $T[\sum_1^n \lambda_i . x_i] = \sum_1^n \lambda_i T[x_i]$ where $\lambda_1, \ldots, \lambda_n \in \mathbb{C}$, $x_1, \ldots, x_n \in X$. The space of all linear functions from $X$ into $\mathbb{K}$ is denoted by $\mathcal{L}(X, \mathbb{K})$ or, more simply, $X'$.

## II.5 Continuous linear functionals on a normed space

Let $(X, \|\cdot\|)$ be a normed space (i.e., a vector space $X$ endowed with a norm). A linear functional $T: X \to \mathbb{K}$ is <u>bounded</u> if $|T(x)| \leq c\|x\|$, $\forall x \in X$, for some constant $C \geq 0$.

<u>Proposition</u>: Given a linear functional $T: X \to \mathbb{K}$, the following are equivalent:
(1) $T$ is bounded
(2) $T$ is continuous
(3) $T$ is continuous at $0$
(4) $T$ is uniformly continuous

Proof: ① $\Rightarrow$ ④

$$|Tx-Ty| = |T(x-y)| \leq c\|x-y\|$$. It follows that $T$ is uniformly continuous.

④ $\Rightarrow$ ② and ② $\Rightarrow$ ③ are obvious.

③ $\Rightarrow$ ① given $\varepsilon>0$, $\exists\, \delta>0$ : $\|x-0\| = \|x\| < \delta \Rightarrow |Tx| < \varepsilon$.

Now, given $x \neq 0$, $\left\| \frac{\delta}{2} \frac{x}{\|x\|} \right\| = \delta/2 < \delta$, so $\left| T\left(\frac{\delta}{2}\frac{x}{\|x\|}\right)\right| < \varepsilon$.

Then $|Tx| < \frac{2}{\delta}\varepsilon . \|x\|$, $\forall x \neq 0$. So $T$ is bounded. □

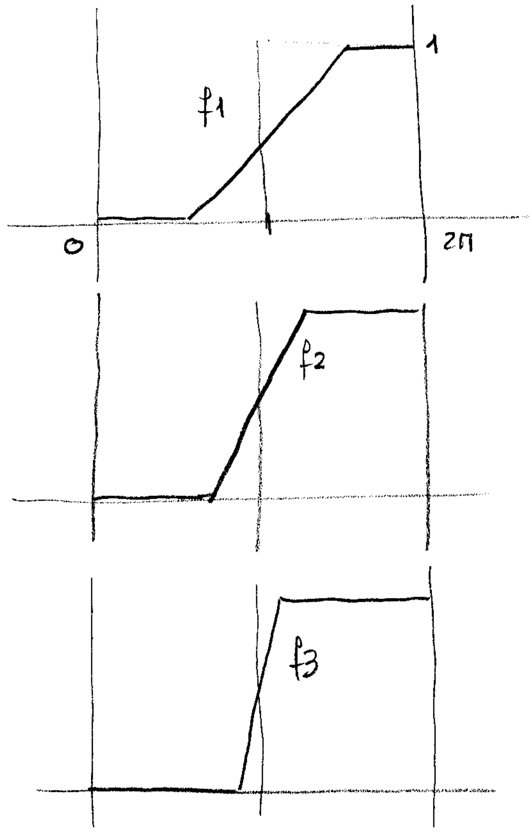
Remark: Observe that the boundedness of a linear functional means that the image of the closed unit ball is a bounded set in $\mathbb{C}$.

Observe that, in case $X$ is finite dimensional, every linear functional is continuous.

## Remark

$(C[0,2\pi], \|\cdot\|_1)$ is not complete:

It is enough to define a Cauchy sequence which does not converge. This is simple: Take $(f_n)$ in the following way:

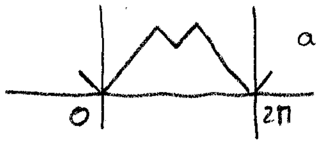



It is clear that
$$\|f_n - f_m\|_1 = \int_0^{2\pi} |f_n(t) - f_m(t)|\,dt$$
goes to 0 as $n, m \to \infty$.
It is then a Cauchy sequence. However, it does not converge in $\|\cdot\|_1$ to any continuous function.
We know, however, that $(C[0,2\pi], \|\cdot\|_\infty)$ is complete. This proves, in particular, that $\|\cdot\|_1$ and $\|\cdot\|_\infty$ are not equivalent (see below)

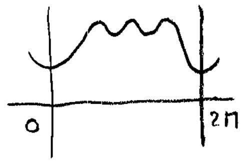
Two norms $\|\cdot\|$ and $|||\cdot|||$ in a normed space $X$ are equivalent if there exists $A, B$ such that $A \neq 0$, $B \neq 0$,
$$A\|x\| \leq |||x||| \leq B\|x\|, \quad \forall x \in X$$
Assume $\|\cdot\| \sim |||\cdot|||$ (i.e., they are equivalent). Let $(x_n)$ be a $\|\cdot\|$-Cauchy sequence in $X$. Then $(x_n)$ is $|||\cdot|||$-Cauchy. If $(X, |||\cdot|||)$ is complete, $x_n \xrightarrow{|||\cdot|||} x$ for some $x \in X$. It follows that $x_n \xrightarrow{\|\cdot\|} x$ and so $(X, \|\cdot\|)$ is complete, too.

## III Smooth periodic functions

III.1 We introduced before $\mathcal{C}$, the space of continuous $2\pi$-periodic functions. We shall introduce now a subspace of this linear space. Let $\mathcal{P}$ be the set of all smooth functions in $\mathcal{C}$ (i.e., the set of all infinitely differentiable functions in $\mathcal{C}$, i.e., $2\pi$-periodic)



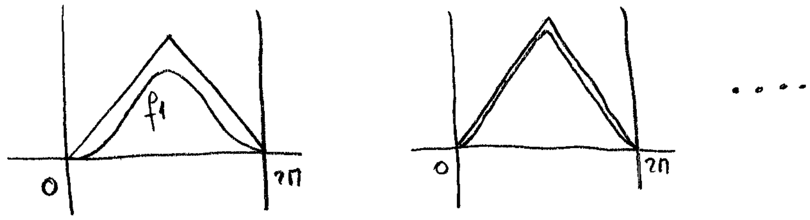

a function in $\mathcal{C}$




a function in $\mathcal{P}$

Remark. If $f \in \mathcal{C}$ and there exists $f'$, it is obvious that $f'$ is also $2\pi$-periodic. In particular, if $f \in \mathcal{P}$ it follows that $f^{(k)} \in \mathcal{P}$, for all $k = 0, 1, 2, \ldots$

Remark $\mathcal{P} \subset \mathcal{C}$ has a serious drawback: it is not complete (Hint: consider the following sequence of element of $\mathcal{P}$:




a precise description will be given later).

To avoid this we introduce a different kind of convergence on $\mathcal{P}$

III.2 A sequence $(f_n)_1^\infty$ in $\mathcal{P}$ converges to $f \in \mathcal{P}$ in the sense of $\mathcal{P}$ (and we shall write $f_n \xrightarrow{\mathcal{P}} f$) whenever

$$D^k f_n \xrightarrow[n \to \infty]{\|\cdot\|_\infty} D^k f \quad , \text{ for } k = 0, 1, 2, \ldots$$

(here $D^k$ stands for the $k$-th derivative)

A sequence $(f_n)$ in $\mathcal{P}$ is Cauchy in the sense of $\mathcal{P}$ ($\mathcal{P}$-Cauchy, in short) whenever $(D^k f_n)_n$ is Cauchy in $(\mathcal{C}, \|\cdot\|_\infty)$, $k = 0, 1, 2, \ldots$

III-3

> **Theorem**: All $\mathcal{P}$-Cauchy sequences in $\mathcal{P}$ are $\mathcal{P}$-convergent to some element in $\mathcal{P}$.

Proof: Let $(f_n)_1^\infty$ be a $\mathcal{P}$-Cauchy sequence in $\mathcal{P}$. Fix $k = 0, 1, 2, \ldots$ Consider $(D^k f_n)_n$, a $\|\cdot\|_\infty$-Cauchy sequence in $\mathcal{C}$. As $(\mathcal{C}, \|\cdot\|_\infty)$ is complete, it $\|\cdot\|_\infty$-converges to some $g_k \in \mathcal{C}$. We shall prove that $g_0 \in \mathcal{P}$ and $D^k g_0 = g_k$, $k = 1, 2, 3, \ldots$ This will prove that $g_0 \in \mathcal{P}$ and $(D^k f_n)_n$ is $\|\cdot\|_\infty$-convergent to $D^k g_0$, $k = 1, 2, \ldots$, i.e., which amounts to say that $(f_n) \xrightarrow{\mathcal{P}} g_0$.

$$D^k f_n(x) = D^k f_n(0) + \int_0^x D^{k+1} f_n(t)\,dt , \quad n=1,2,3\dots , \quad x \in \mathbb{R}$$

Letting $n \to \infty$ (for $x$ fixed and $k$ fixed) we get

$$g_k(x) = g_k(0) + \int_0^x g_{k+1}(t)\,dt \qquad [**]$$

(the last element because the convergence of $D^{k+1} f_n$ to $g_{k+1}$ is uniform). $[**]$ implies that $D g_k = g_{k+1}$. By induction we get $D^k g_0 = g_k$, $k=1,2,3\dots$ □

**III.4** <u>Remark</u> on $\mathcal{P}$, a metric can be defined in such a way that convergence with respect to this metric is the same as what has been defined as $\mathcal{P}$-convergence. Just define

$$\left\langle d(f,g) \equiv \sum_{k=0}^{\infty} \frac{1}{2^{k+1}} \frac{\| D^k f - D^k g \|_\infty}{1 + \| D^k f - D^k g \|_\infty} \right\rangle$$

To prove this assertion, follow the hint below.

(Hint: Given a sequence $(f_n)$ in $\mathcal{P}$, $f_n \xrightarrow{\mathcal{P}} f \in \mathcal{P}$, we know $\| D^k f_n - D^k f \|_\infty \xrightarrow[n\to\infty]{} 0$. Given $\varepsilon>0$, find $k_0 \in \mathbb{N}$ such that $\sum_{k_0+1}^{\infty} \frac{1}{2^{k+1}} < \varepsilon/2$. Find $n_0 \in \mathbb{N}$ such that $\| D^k f_n - D^k f \|_\infty < \frac{\varepsilon}{(k_0+1)2}$ for $k = 0,1,2,\dots k_0$. It follows that $d(f_n, f) < \varepsilon$, $\forall n \geq n_0$.
To prove the converse, proceed in a similar (more simple) way.

**III.5** <u>Remark</u> : On $\mathcal{P}$, the function $f \xrightarrow{S_k} |D^k f|$ , $k = 0,1,2,\dots$ is an example of a <u>seminorm</u>, i.e., a function similar to a norm except that $S_k(f)=0$ does not necessarily implies $f=0$.
$\mathcal{P}$ has then a sequence of seminorms $(S_k)_0^\infty$ defining the convergence, and $\mathcal{P}$ is complete (Theorem III.3). This situation is described by the term <u>Fréchet space</u>.

$\mathcal{P}$ with this sort of convergence <u>is a Fréchet space</u>.
The convergence is given by the distance $d(f,g) := \sum_{k=0}^{\infty} \frac{1}{2^{k+1}} \frac{\|f-g\|_k}{1+\|f-g\|_k}$

III.6. A very interesting fact is that $\mathcal{P}$ has a convergence structure (the one defined by $d$) which cannot be described by a norm. Then the sequence of seminorms given in III.5 is really necessary.

The following argument applies in many other circumstances:

1) We shall prove that, in the case a certain norm $|||\cdot|||$ on $\mathcal{P}$ gives the same convergence as $d$, there exists $M>0$ and $N\geq 0$ such that

$$\boxed{|||f||| \leq M\{\|f\|_0 + \dots + \|f\|_N\}\ ,\ \forall f\in\mathcal{P}} \qquad (*)$$

Assume $(*)$ is not true. Then $\forall M>0$ and $N\geq 0$ some $f$ violates $(*)$. In particular, given $n\in\{0,1,2,\dots\}$ we can find $f_n\in\mathcal{P}$ such that $|||f_n||| > n\{\|f_n\|_0+\dots+\|f_n\|_n\} \geq n\|f_n\|_k$, $k=0,\dots,n$

Then
$$\left\|\frac{f_n}{|||f_n|||}\right\|_k = \frac{1}{|||f_n|||}\cdot\|f_n\|_k \leq \frac{1}{n}, \quad k\leq n$$

Letting $n\to+\infty$ we get that $\|f_n/|||f_n|||\|_k \to 0$. As this is true for every $k$, it follows that

$$f_n/|||f_n||| \xrightarrow{\mathcal{P}} 0$$

However $|||f_n/|||f_n||| ||| = 1$, $\forall n$, and we get a contradiction. So $(*)$ is true. □

2) Pick $f_n(x) = \dfrac{\operatorname{sen}(nx)}{n^{N+1}}$, $x\in\mathbb{R}$, $n\in\mathbb{N}$.

Then $f_n\in\mathcal{P}$, $\forall n$.

$$|D^k f_n(x)| = \frac{1}{n^{N-k+1}}, \quad k=0,1,2,\dots N,\ x\in\mathbb{R}$$

Then $\|f_n\|_k \xrightarrow[n\to\infty]{} 0$, $k=0,1,2,\dots,N$.

From $(*)$ it follows that $|||f_n|||\to 0$

Observe that

$$\|D^{N+1}f_n\|_\infty = 1, \text{ then } \|f_n\|_{N+1} = 1, \forall n$$

and so $f_n \overset{\mathcal{P}}{\not\to} 0$, a contradiction. □

Exercises

(1) Prove that every linear functional on a finite dimensional normed space is continuous (Hint: a finite dimensional normed space is $\mathbb{R}^n$ or $\mathbb{C}^n$. They have a finite algebraic basis. Write the general form of a linear functional on $\mathbb{R}^n$ or $\mathbb{C}^n$).

(2) Define, given $x \in \mathbb{R}$, a linear functional on $(\mathcal{C}, \|\cdot\|_\infty)$ as $f \to f(x)$. Prove that in fact is linear and, moreover, is continuous.

(3) Define on $\mathcal{C}$ the following function

$$\operatorname{int}(f) = \frac{1}{2\pi} \int_0^{2\pi} |f(x)| dx$$

a) Prove that $\operatorname{int}()$ is a norm on $\mathcal{C}$.

b) Prove that $\operatorname{int}(f) \leq \|f\|_\infty$, $\forall f \in \mathcal{C}$.

c) Prove that $(\mathcal{C}, \operatorname{int}(\cdot))$ is a normed space, but it is not complete

(4) Define $l^1(\mathbb{Z}) \equiv \{ (a_n)_{n \in \mathbb{Z}}, \sum_{-\infty}^{+\infty} |a_n| < \infty \}$ where $\sum_{-\infty}^{+\infty} |a_n| = \sum_0^{+\infty} |a_n| + \sum_1^{\infty} |a_{-n}|$. Endow $l^1(\mathbb{Z})$ with the norm $\|(a_n)\|_1 = \sum_{\mathbb{Z}} |a_n|$. Prove that $l^1(\mathbb{Z})$ is a Banach space.

Define $f: \mathbb{R} \to \mathbb{C}$ as $f(x) \equiv \sum_{\mathbb{Z}} a_n e^{inx}$, $x \in \mathbb{R}$.

Prove that $f \in \mathcal{C}$

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(5) Give a precise sequence of functions in $\mathcal{P}$, Cauchy in $(\mathcal{C}, \|\cdot\|_\infty)$, but not convergent to an element in $\mathcal{P}$.

(6) Prove the remaining part of Remark III.4

(7) Using I.3 give explicit examples of functions in $\mathcal{P}$.

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(8) Decide if the following functions on $(\mathcal{C}, \|\cdot\|_\infty)$ are linear and if they are continuous

a) $T(f) = f(\pi/2)$

b) $T(f) = \int_0^{2\pi} f(x) \operatorname{sen} nx \, dx$

c) $T(f) = \int_0^{2\pi} f^2(x) dx$

d) $T(f) = f(0) + \int_\pi^{2\pi} f(x) dx$

e) $T(f) = |f(0)|$

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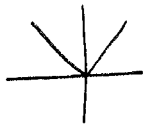
(9) Are the following sequences $\mathcal{P}$-Cauchy in $\mathcal{P}$?

a) $f_n(x) = n^3 \cos nx$, b) $f_n(x) = \frac{\operatorname{sen} nx}{n!}$, c) $f_n(x) = \sum_{m=1}^{n} \frac{\operatorname{sen}(mx)}{m!}$

# IV $\boxed{\text{Approximation of continuous periodic functions by smooth periodic ones}}$

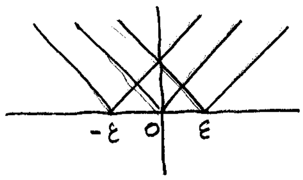
**IV.1.** Consider a continuous function on $\mathbb{R}$, say $f$. Of course, $f$ is not necessarily differentiable. An example is the modulus function

$$f(x) = |x|, \ \forall x \in \mathbb{R}$$



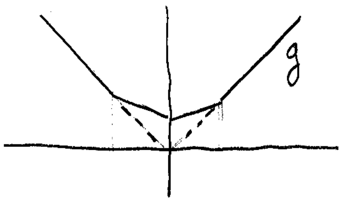
This function is not differentiable at $0$. However, an average of translates of $f$ is, in a sense, «smoother» than $f$, and, if the translates are translated slightly, the average is not far from $f$. For example, take translations of magnitude $\varepsilon$ :

$$|x-\varepsilon|, \ |x|, \ |x+\varepsilon|$$

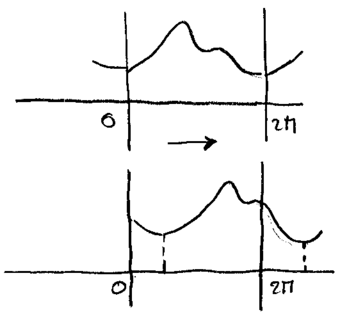



and form the average

$$\frac{1}{3}|x-\varepsilon| + \frac{1}{3}|x| + \frac{1}{3}|x+\varepsilon| = g(x)$$




Define $T_t f(x) = f(x-t)$, $x \in \mathbb{R}$ for some $t$, where $f \in \mathcal{C}$ : this is a shifted function.




It is again $2\pi$-periodic. Note that $t \approx 0$ or $t \approx 2\pi$ produce a very small shift.
Now, take

$$0 \leq t_0 < t_1 < \ldots < t_k \leq 2\pi$$

We want an average of shifted function, and this means

$$a_0 T_{t_0} f + a_1 T_{t_1} f + \cdots + a_k T_{t_k} f = g$$

where $a_0 + a_1 + a_2 + \cdots + a_k = 1$,
$a_i \geqslant 0$, $i = 1, 2, \ldots, k$.

If we write
$$\frac{a_r}{t_r - t_{r-1}} = b(t_r), \quad r = 1, 2, \ldots, k, \quad \frac{a_0}{t_0 - t_{-1}} = b(t_0), \quad t_{-1} = t_k - 2\pi,$$

we get
$$g = b(t_0) \cdot T_{t_0} f \cdot (t_0 - t_{-1}) + b(t_1) T_{t_1} f \cdot (t_1 - t_0) + \cdots$$
$$\cdots + b(t_k) \cdot T_{t_k} f (t_k - t_{k-1}).$$

IV.2. This suggests defining

$$g(x) \equiv \int_0^{2\pi} b(t) \cdot T_t f(x)\, dt = \int_0^{2\pi} b(t)\, f(x-t)\, dt$$

Observe that $b$ is a function playing the rôle of assigning weights (in fact, distributing the weight 1 along the interval $[0,2\pi]$).

Let's introduce the operation

$$b \in \mathcal{C},\ f \in \mathcal{C} \Rightarrow (b * f)(x) = \frac{1}{2\pi}\int_0^{2\pi} b(x-t)\, f(t)\, dt =$$

$(x-t=y \Rightarrow t = x-y \Rightarrow dt = -dy)$

$$= \frac{1}{2\pi}\int_x^{x-2\pi} b(y)\, f(x-y)\,(-dy) = \frac{1}{2\pi}\int_0^{2\pi} b(t)\, f(x-t)\, dt$$

by periodicity. It is called the <u>convolution</u> of $b$ and $f$

IV.3

<u>Proposition</u>

1. $f, g \in \mathcal{C} \Rightarrow f*g \in \mathcal{C}$
2. $f*g = g*f$
3. $(\alpha f)*g = \alpha(f*g)$
4. $(f+g)*h = (f*h)+(g*h)$
5. $(f*g)*h = f*(g*h)$
6. $T_t(f*g) = (T_t f)*g = f*T_t g$

<u>Proof</u>: Observe first that the definition of convolution makes sense:

$f \in \mathcal{C}, g \in \mathcal{C} \Rightarrow$ for $x \in \mathbb{R}$, $t \to f(x-t)\,g(t)$ is a continuous function, so there exists $\int_0^{2\pi} f(x-t)\,g(t)\,dt$ ; moreover

$$\left|(f*g)(x)\right| = \left|\int_0^{2\pi} f(x-t)\, g(t)\,dt\right| \frac{1}{2\pi} \leq \|f\|_\infty \cdot \|g\|_\infty$$

and so $f*g$ is a bounded function. It is obvious that it is also $2\pi$-periodic. It follows that

$$\|f*g\|_\infty \leq \|f\|_\infty \cdot \|g\|_\infty \qquad [*]$$

All statements 2 to 6 are straightforward. Let us prove that $(f*g)$ is continuous:

$$\|T_\varepsilon(f*g) - (f*g)\|_\infty = \|(T_\varepsilon f - f)*g\|_\infty \leq \|T_\varepsilon f - f\|_\infty \|g\|_\infty \qquad [**]$$

$f$ is continuous in $[0,2\pi]$, so it is uniformly continuous (Heine-Borel Theorem) and, as it is periodic, it is uniformly continuous in $\mathbb{R}$.

It follows that, if $\varepsilon > 0$ is small, $\|T_\varepsilon f - f\|_\infty$ is small, and then $\|T_\varepsilon(f*g) - (f*g)\|_\infty$ is also small. This means that $(f*g)$ is continuous. □

IV.4 <u>Remark</u>: It is worth mentioning that inequality [**] says that it is enough that ONE of the functions (in this case $f$) should be continuous in order to get continuity of $(f*g)$ (in any case we need to be able to integrate). The conclusion is that ONE good function ($f$ or $g$) makes $f*g$ good also, although the other was not so good. This idea appears below again.

Given $f\in\mathcal{C}$, consider, given $t\in\mathbb{R}$, $t\neq 0$

$$x \xrightarrow[Q_t]{} \frac{f(x+t)-f(x)}{t} \qquad \text{(difference quotient)}$$

As $t$ runs in $\mathbb{R}$, $t\neq 0$, we get a family of functions $\{Q_t\}_{t\neq 0}$
Observe that all of them are continuous and $2\pi$-periodic.

IV.5 <u>Proposition</u>

$$\boxed{\text{In case } f\in\mathcal{P}, \quad Q_t \xrightarrow[t\to 0]{\mathcal{P}} Df}$$

<u>Proof</u>: First step: $Q_t \xrightarrow[t\to 0]{\|\cdot\|_\infty} Df$

Given $x\in\mathbb{R}$, use the Mean Value Theorem to get

$$|Q_t(x)-Df(x)| = |Df(x')-Df(x)|$$

where $x'\in[x, x+t]$. As $Df$ is uniformly continuous in $\mathbb{R}$, we get the conclusion.

Note now that

$$D(T_{-t}f)(x) = \lim_{\varepsilon\to 0}\frac{T_{-t}f(x+\varepsilon)-T_{-t}f(x)}{\varepsilon} =$$

$$=\lim_{\varepsilon\to 0}\frac{f(x+t+\varepsilon)-f(x+t)}{\varepsilon} = Df(x+t) = (T_{-t}Df)(x)$$

Analogously, $\quad D^k(T_{-t}f) = T_{-t}D^k f$

Then $\quad D^kQ_t = D^k\left[(T_{-t}f-f)\frac{1}{t}\right] = \left(T_{-t}D^kf - D^kf\right)\frac{1}{t}$

$$= Q_t[D^kf]$$

$$\|D^kQ_t - D^{k+1}f\|_\infty = \|Q_t[D^kf]-D^{k+1}f\|_\infty \xrightarrow[t\to 0]{} 0$$

as it follows from the first step. $\square$

The following Proposition insists in what has been said in the Remark: May be $g$ is just continuous; however, if $f \in \mathcal{P}$, then $f*g \in \mathcal{P}$.

IV.6 $\boxed{\text{Proposition} \quad f \in \mathcal{P},\ g \in \mathcal{C} \Rightarrow f*g \in \mathcal{P} \text{ and } D^k(f*g) = (D^k f)*g}$

Proof: $\frac{1}{t}\left[T_{-t}(f*g) - (f*g)\right] = \frac{1}{t}\left[(T_{-t}f)*g - (f*g)\right] =$

$= \frac{1}{t}\left[(T_{-t}f - f)*g\right] = \left[\frac{1}{t}(T_{-t}f - f)\right]*g = Q_t * g$

We know (IV.5) that $Q_t \xrightarrow{\|\cdot\|_\infty} Df$ and that (see (*) in page 11)

$$\| Q_t * g - Df * g \|_\infty = \| (Q_t - Df) * g \|_\infty \leq \| Q_t - Df \|_\infty \cdot \| g \|_\infty \to 0$$

as $t \to 0$. We get then $\exists\, D(f*g)$ and that $D(f*g) = Df * g$.
Now we proceed by induction $\square$

IV.7 Corollary

$\boxed{\text{Let } (g_n) \in \mathcal{C},\ g_n \xrightarrow{\|\cdot\|_\infty} g \in \mathcal{C}. \text{ Then, given } f \in \mathcal{P},\ f*g_n \xrightarrow{\mathcal{P}} f*g}$

Proof: $\| D^k(f*g_n) - D^k(f*g) \|_\infty = \| D^k f * g_n - D^k f * g \|_\infty =$

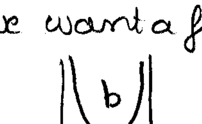
$= \| D^k f * (g_n - g) \| \leq \| D^k f \|_\infty \| g_n - g \|_\infty \xrightarrow[n\to\infty]{} 0$. $\square$

Approximation

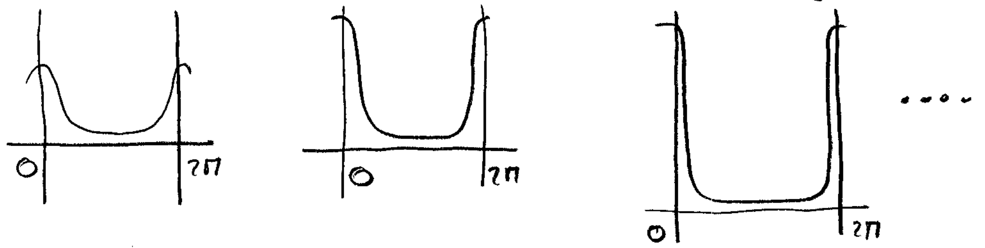
IV.8 Now we return to the problem of approximating continuous periodic functions by smooth periodic functions.
Recall that we wanted small shifts and averages of those shifts. In the discrete case

$$a_0 T_{t_0} f + a_1 T_{t_1} f + \dots + a_k T_{t_k} f, \quad 0 \leq t_0 < \dots < t_k \leq 2\pi$$

where $a_i \geq 0, \forall i$, $\Sigma a_i = 1$. The closer the t's to 0 and $2\pi$, the better. Another way to achieve it is to let $0 \leq t_0 < \dots < t_k \leq 2\pi$ and make $a_0$ and $a_k$ substantially big and the rest substantially small. In the continuos case we want a function $b(t)$ asignating weight 1 as follows



In fact, we want better and better « approximating weights »:




IV.9

$(\varphi_n)_1^\infty$ in $\mathcal{E}$ is an <u>approximate identity</u> whenever

1. $\varphi_n(x) \geqslant 0$, $\forall n, \forall x$
2. $\frac{1}{2\pi}\int_0^{2\pi} \varphi_n(x)\,dx = 1$, $\forall n$
3. $\forall\, 0<\delta<\pi$, $\int_\delta^{2\pi-\delta} \varphi_n(x)\,dx \xrightarrow[n]{} 0$

Remark: all together means that the weights accumulate near 0 and $2\pi$.

IV.10

<u>Theorem</u> Let $(\varphi_n) \subset \mathcal{E}$ be an approximate identity. Then, given $f \in \mathcal{E}$

$$\varphi_n * f \xrightarrow{\|\cdot\|_\infty} f.$$

If $f \in \mathcal{P}$, then $\varphi_n * f \xrightarrow{\mathcal{P}} f$

<u>Proof</u>: It consists in an easy computation: Fix $x \in \mathbb{R}$ and fix $\delta > 0$.

$$2\pi \left| (\varphi_n * f)(x) - f(x) \right| = \left| \int_0^{2\pi} \varphi_n(y) \cdot f(x-y)\,dy - \int_0^{2\pi} \varphi_n(y) f(x)\,dy \right| =$$

$$= \left| \int_0^{2\pi} \varphi_n(y)\left[ f(x-y) - f(x) \right] dy \right| \leqslant \int_0^{2\pi} \varphi_n(y) \left| f(x-y) - f(x) \right| dy =$$

$$= \underset{(1)}{\int_0^\delta} + \underset{(2)}{\int_\delta^{2\pi-\delta}} + \underset{(3)}{\int_{2\pi-\delta}^{2\pi}}$$

(Now the argument is that integrals (1) and (3) are small because of $(T_y f - f)$, and integral (2) is small because of $\varphi$)

$$\leqslant 2 \sup_{|y|\leqslant\delta} \| T_y f - f \|_\infty + 2\|f\|_\infty \int_\delta^{2\pi-\delta} \varphi_n(y)\,dy$$

As $f$ is uniformly continuous, we can find, given $\varepsilon>0$, $\delta>0$ such that $\sup_{|y|\leqslant\delta} \| T_y f - f \|_\infty < \varepsilon$. Then we can find $n_0 \in \mathbb{N}$ such that $\int_\delta^{2\pi-\delta} \varphi_n(y)\,dy < \varepsilon$ for $n \geqslant n_0$. We get then, for $n \geqslant n_0$,

$$\| \varphi_n * f - f \|_\infty \leqslant \frac{1}{2\pi}\left[ 2\varepsilon + 2\|f\|_\infty \varepsilon \right].$$

as it was to be proved.

Now, $D^k(\varphi_n * f) - D^k f = \varphi_n * D^k f - D^k f$, if $f \in \mathcal{P}$.

By the first part $\varphi_n * D^k f \xrightarrow[n]{\|\cdot\|_\infty} D^k f$, for any $k$. It follows that

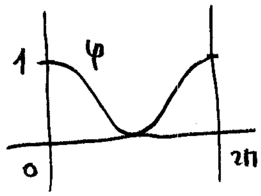
$$\varphi_n * f \xrightarrow[\mathcal{P}]{} f. \qquad \square$$

IV.11 We shall construct a simple sequence which is an approximate identity; it consists of trigonometric polynomials: A trigonometric polynomial is a (complex valued) function

$$\varphi(x) = \sum_{-n}^{n} a_k e^{ikx}$$

Note that those are the partial sums of a trigonometric Series, used in the theory of Fourier series.

Let $\varphi(x) = \frac{1}{2}(1+\cos x)$




It is quite clear that successive powers of this function (corrected in order to get ② in IV.9) behaves as it was expected. Precisely, set

$$\left\langle \varphi_n(x) = c_n(1+\cos x)^n, \quad n \in \mathbb{N} \right\rangle$$

where $c_n > 0$ is such that $\frac{1}{2\pi}\int_0^{2\pi} \varphi_n(x)\,dx = 1$.

It is a simple exercise to prove that the sequence $(\varphi_n)_1^\infty$ is an approximate identity. Observe that $\varphi_n \in \mathcal{P}$, $\forall n$. It is simple, too, to prove that $\varphi_n * f$ is a trigonometric polynomial for any $f \in \mathcal{C}$. We get then, as a Corollary of IV.10, the famous

IV.12 Corollary [The Weierstrass Approximation Theorem]: The trigonometric polynomial are dense in the space of $2\pi$-periodic continuous functions $\mathcal{C}$ and in the space of $2\pi$-periodic smooth functions $\mathcal{P}$

Proof: Given $f \in \mathcal{C}$, $(\varphi_n * f) \xrightarrow{\|\cdot\|_\infty} f$, and $\varphi_n * f$ are trigonometric polinomials. Given $f \in \mathcal{P}$, $(\varphi_n * f) \xrightarrow{\mathcal{P}} f$. □

IV.13 Remark: The method used gives not only a theoretical (existence) result, but an actual sequence of approximating functions.

IV.14 Remark: A trigonometric polynomial can be approximated as much as we wish by a polynomial (use de Taylor expansion) (uniformly), so we get that there is a sequence of polynomials which converge uniformly to $f \in \mathcal{C}$. Note also that $[0, 2\pi]$ can be replaced by any other finite interval in $\mathbb{R}$. We can then formulate the classical Weierstrass Approximation Theorem: any continuous function in $[a,b]$ is the uniform limit of a sequence of polynomials.

Exercises :

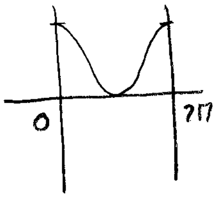
① Prove that the sequence $(\varphi_n)$ given in IV.11 is actually an approximate identity.

② Prove that any trigonometric polynomial can be approximate in $[0,2\pi]$ as much as we wish by a polynomial (uniformly in $[0,2\pi]$).

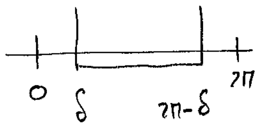
③ Define $e_k(x) = \exp(ikx)$, $x \in \mathbb{R}$, $k \in \mathbb{Z}$. Prove that those are functions in $\mathcal{P}$. Given $f \in \mathcal{C}$, compute $e_k * f$.

④ Compute $e_k * e_j$ for any two functions in exercise ③.

(1) $\varphi_n(x) = c_n\left(\frac{1+\cos x}{2}\right)^n$, $\frac{1}{2\pi}\int_0^{2\pi}\varphi_n(x)\,dx = 1$

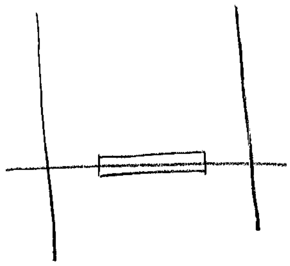



Fix $\delta > 0$




$$\frac{1}{2\pi}\cdot c_n\int_0^{2\pi}\left(\frac{1+\cos x}{2}\right)^n dx = 1$$

$$\frac{1}{2\pi}\int_\delta^{2\pi-\delta}\varphi_n(x)\,dx = \frac{1}{2\pi}c_n\int_\delta^{2\pi-\delta}\left(\frac{1+\cos x}{2}\right)^n dx = \left(\int_0^{2\pi}\left(\frac{1+\cos x}{2}\right)^n dx\right)^{-1}\cdot\int_\delta^{2\pi-\delta}\left(\frac{1+\cos x}{2}\right)^n dx$$



$1+\cos x < r(1+\cos y)$
$x \in [\delta, 2\pi-\delta]$, $y \in [0, \delta/2]$
for some $r \in \,]0,1[$

$$\varphi_n(x) = c_n(1+\cos x)^n < c_n r^n (1+\cos y)^n = r^n\varphi_n(y), \quad x\in[\delta,2\pi-\delta],\ y\in[0,\delta/2]$$

Integrating in $y \in [0, \tfrac{1}{2}\delta]$,

$$\varphi_n(x)\int_0^{\delta/2} dy \le r^n\int_0^{\delta/2}\varphi_n(y)\,dy \le 2\pi r^n$$

$$\frac{1}{2}\delta\cdot\varphi_n(x) \le 2\pi r^n$$

$$\varphi_n(x) \le 4\pi\delta^{-1} r^n, \quad x\in[\delta,2\pi-\delta]$$

$\Rightarrow \varphi_n \longrightarrow 0$ uniformly in $[\delta, 2\pi-\delta]$

(3) $e_k(x) = \exp(ikx)$, $x\in\mathbb{R}$, $k\in\mathbb{Z}$
$De_k(x) = ik\exp(ikx) = ik\cdot e_k(x)$
$D^2 e_k(x) = (ik)^2 e_k(x)$
$D^m e_k(x) = (ik)^m\cdot e_k(x)$

$e_k \in \mathcal{P}$

$$f\in\mathcal{C} \Rightarrow (e_k * f)(x) = \frac{1}{2\pi}\int_0^{2\pi} e_k(x-y)f(y)\,dy = \frac{1}{2\pi}\int_0^{2\pi} e^{ikx}\cdot e^{-iky}f(y)\,dy =$$

$$= e^{ikx}\left(\frac{1}{2\pi}\int_0^{2\pi} e^{-iky}f(y)\,dy\right)$$ : k-th term of the Fourier series.

$$(e_k * e_j)(x) = \frac{1}{2\pi}\int_0^{2\pi} e_k(x-y)\, e_j(y)\, dy = e^{ikx}\left(\frac{1}{2\pi}\int_0^{2\pi} e^{-iky} \cdot e^{ijy}\, dy\right)$$

$$= e^{ikx}\left(\frac{1}{2\pi}\int_0^{2\pi} e^{i(j-k)y}\, dy\right) = \begin{cases} e^{ikx}, & \text{if } k=j \\ 0, & \text{if } k\neq j \end{cases}$$

$$\int_0^{2\pi} e^{ipx}\, dx = \int_0^{2\pi} \cos px\, dx + i\int_0^{2\pi} \sin px\, dx = 0$$

## V Periodic distributions

Let's recall some basic facts:

$\mathcal{P}$ is the space of all infinitely differentiable $2\pi$-periodic functions (with real or complex values).

$\mathcal{P}$ has not a norm to describe the convergence on $\mathcal{P}$. Instead, it has a sequence of «seminorms» given by

$$f \longrightarrow \|D^k f\|_\infty \quad , \; k=0,1,2,\cdots \qquad (*)$$

in the sense that a sequence $(f_n)_1^\infty$ in $\mathcal{P}$ converges to $f\in\mathcal{P}$ in the sense of $\mathcal{P}$ (and we write $(f_n) \xrightarrow[n\to\infty]{\mathcal{P}} f$ )

whenever $\quad D^k f_n \xrightarrow[n\to\infty]{\|\,\|_\infty} D^k f \quad , \; k=0,1,2,\cdots$

Alternatively, a distance $d$ can be introduced in such a way that

$$f_n \xrightarrow[n\to\infty]{\mathcal{P}} f \;\equiv\; d(f-f_n) \xrightarrow[n\to\infty]{} 0$$

This distance is

$$d(f,g) \equiv \sum_{k=0}^{\infty} \frac{\|D^k f - D^k g\|_\infty}{1+\|D^k f - D^k g\|_\infty} \qquad (**)$$

V.1 <u>Definition</u> A <u>periodic distribution</u> $F$ is a continuous linear mapping on the space $\mathcal{P}$ endowed with the convergence defined by $d$ given in $(**)$ (or by the sequence of seminorms given in $(*)$). Precisely

$$\mathcal{P} \xrightarrow[\text{linear}]{F} \mathbb{C}$$

and if $f_n \xrightarrow{\mathcal{P}} f$ then $F(f_n) \longrightarrow F(f)$

The space of all periodic distributions is denoted by $\mathcal{P}'$

V.2 <u>Some elementary examples of periodic distributions</u>

1. Fix $x_0\in\mathbb{R}$. Define $F:\mathcal{P}\to\mathbb{C}$ by $F(f)=f(x_0)$, $\forall f\in\mathcal{P}$. Then $F$ is a periodic distribution. Those distributions are called <u>Dirac delta's</u>, and are denoted by $\delta_{x_0}$.

To prove that $\delta_{x_0}$ is in fact an element in $\mathcal{P}'$ we need to prove first that it is linear: For $f, g \in \mathcal{P}$, $\alpha \in \mathbb{C}$,

$$\delta_{x_0}(f+g) = (f+g)(x_0) = f(x_0) + g(x_0) = \delta_{x_0}(f) + \delta_{x_0}(g)$$
$$\delta_{x_0}(\alpha f) = (\alpha f)(x_0) = \alpha f(x_0) = \alpha\, \delta_{x_0}(f)$$

and that it is continuous:

Let $(f_n)$ a sequence in $\mathcal{P}$, $f_n \xrightarrow{\mathcal{P}} f \in \mathcal{P}$. In particular $f_n \xrightarrow{\|\cdot\|_\infty} f$, so $f_n(x_0) \xrightarrow[n]{} f(x_0)$. So we get

$$\delta_{x_0}(f_n) \xrightarrow[n]{} \delta_{x_0}(f).$$

(2) Fix $v \in \mathcal{C}$. Define $F: \mathcal{P} \to \mathbb{C}$ by

$$F(f) := \frac{1}{2\pi} \int_0^{2\pi} v(x) \cdot f(x)\, dx \ , \ \forall f \in \mathcal{P}.$$

Then $F$ is a periodic distribution. Those distributions are called <u>functions</u> and are denoted $F_v$.

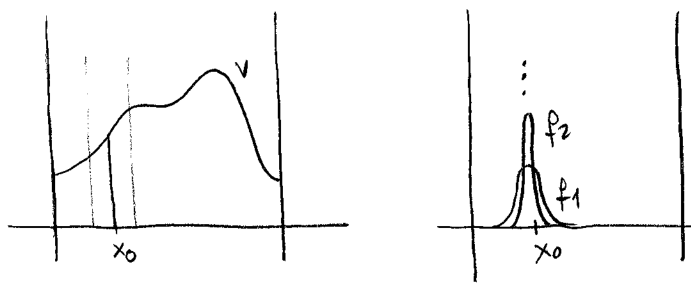
Again, to prove that in fact $F_v$ is a periodic distribution, note that $F_v$ is obviously a linear mapping from $\mathcal{P}$ into $\mathbb{C}$.

Now suppose $(f_n)$ in $\mathcal{P}$ satisfies $f_n \xrightarrow[n\to\infty]{\mathcal{P}} f \in \mathcal{P}$. In particular $f_n \xrightarrow[n]{\|\cdot\|_\infty} f$

Then $$\left| F_v(f) - F_v(f_n) \right| = \frac{1}{2\pi} \left| \int_0^{2\pi} v(x) \left[ f(x) - f_n(x) \right] dx \right| \leqslant$$
$$\leqslant \frac{1}{2\pi} \|f - f_n\|_\infty \left| \int_0^{2\pi} v(x)\, dx \right| \xrightarrow[n\to\infty]{} 0$$

It follows that $F_v$ is continuous on $\mathcal{P}$.

5.3 <u>Remark</u>: Given $\delta_{x_0} \in \mathcal{P}'$, it is <u>not</u> a function. To prove that, assume $\delta_{x_0} = F_v$ for some $v \in \mathcal{C}$. Consider a sequence $(f_n)$ in $\mathcal{P}$ like this one:




The support is very small and $f_n(x_0) \to +\infty$
The support is so small that
$$\int_0^{2\pi} f_n(x)\, v(x)\, dx \xrightarrow[n\to\infty]{} 0$$

Then we get $$\left.\begin{array}{l} F_v(f_n) \to 0 \\ \delta_{x_0}(f_n) \longrightarrow +\infty \end{array}\right\} \text{a contradiction.}$$

We shall provide some more examples later on.

V.4 Convergence in the space $\mathcal{P}'$

Given a sequence in $\mathcal{P}'$, say $(F_n)_1^\infty$, and an element $F \in \mathcal{P}'$, we say

$$F_n \xrightarrow[n\to\infty]{\mathcal{P}'} F$$

if $F_n(f) \xrightarrow[n]{} F(f), \forall f \in \mathcal{P}$

This kind of convergence is sometimes called "weak convergence".

V.5 The situation now is the following : we have a linear space $\mathcal{P}'$ (the periodic distributions) with a certain convergence on it. To a periodic continuous function $V \in \mathcal{C}$ we associated a certain element $F_V$ in $\mathcal{P}'$.

We shall prove first that this association is injective, i.e., two different functions $V, W$ in $\mathcal{C}$ produce two different distributions $F_V, F_W$.

We provide here two different proofs of this fact. Both use the Weierstrass Approximation Theorem : Start by assuming

$$F_V = F_W$$

a) In particular $F_V(x^n) = F_W(x^n)$, $\forall n = 0,1,2,\dots$ Then

$$\int_0^{2\pi} [V(x) - W(x)] \cdot x^n \, dx = 0, \quad n = 0,1,2,\dots$$

We shall prove that $\int_0^{2\pi} h(x) x^n dx = 0, \forall n \Rightarrow h \equiv 0$, where $h$ is a continuous function.

$$\int_0^{2\pi} h^2(x)\, dx = \int_0^{2\pi} h(x)[h(x) - P(x)]\, dx, \quad \forall P \text{ polinomial in } [0,2\pi]$$

By the Weierstrass Approximation Theorem, the second member can be made as small as we wish. It follows that $\int_0^{2\pi} h^2(x) dx = 0$. As $h$ is continuous and $h^2 \geq 0$, we get $h^2(x) = 0$, $\forall x \in [0,2\pi]$, so $h(x) = 0$, $\forall x \in [0,2\pi]$.

b) We proved that $\mathcal{P}$ is $\|\cdot\|_\infty$-dense in $\mathcal{C}$. Consider $(V-W)^*$, where $*$ denotes the complex conjugate. There exists a sequence $(\varphi_n)_1^\infty$ in $\mathcal{P}$ such that $\varphi_n \xrightarrow{\|\cdot\|_\infty} (V-W)^*$. Then

$$0 = (F_V - F_W)(\varphi_n) = \frac{1}{2\pi}\int_0^{2\pi} \{V(x) - W(x)\}\, \varphi_n(x)\, dx \xrightarrow[n\to\infty]{} \frac{1}{2\pi}\int_0^{2\pi} |\{V(x) - W(x)\}|^2 dx$$

and then $V(x) = W(x), \forall x \in [0,2\pi]$. □

So we can «identify» continuous periodic functions with some distributions, precisely $V \to F_V$. Those distributions are called, by a slight notational abuse, «functions», and we can think of $\mathcal{E}$ as a subspace of $\mathcal{P}'$.

$\mathcal{E}$ has some natural operations defined on it: sum, product by an scalar, complex conjugate, reversal, translation, ...
The goal is to define appropriately corresponding operations on $\mathcal{P}'$ which extend the ones in $\mathcal{E}$. The main idea is that the action is transferred to $\mathcal{P}$. And then it will happen that even derivatives can be defined on $\mathcal{P}'$ (hence on $\mathcal{E} \subset \mathcal{P}'$) although $f \in \mathcal{E}$ was not differentiable. It will then be possible to "derivate" non differentiable functions and solve some differential equations in cases where no «classical» calculus is available.

## V.6 Operations in $\mathcal{P}'$

1. Complex conjugacy : given $F \in \mathcal{P}'$, define

$$\left\langle F^*(f) := \left[F(f^*)\right]^*, \ \forall f \in \mathcal{P} \right\rangle$$

Remark: this definition agrees with the classical one on functions: Let $V \in \mathcal{E}$. $F_{V^*}(f) = \frac{1}{2\pi}\int_0^{2\pi} V(x)^* f(x)\,dx = \left[\frac{1}{2\pi}\int_0^{2\pi} V(x) \cdot f^*(x)\,dx\right]^* =$

$$= \left[F_V(f^*)\right]^* = F_V^*(f), \ \forall f \in \mathcal{P}. \text{ It follows that}$$

$$F_V^* = F_{V^*}, \text{ for } V \in \mathcal{E}.$$

2. Reversal : given a function $f \in \mathcal{E}$, define $\tilde{f}(x) := f(-x)$, $\forall x \in \mathbb{R}$.
Define now the reversal for a distribution $F \in \mathcal{P}'$:

$$\left\langle \tilde{F}(f) = F(\tilde{f}), \ \forall f \in \mathcal{P} \right\rangle$$

Remark: Again, it agrees with the definition for functions

$$F_{\tilde{V}}(f) = \frac{1}{2\pi}\int_0^{2\pi} \tilde{V}(x) f(x)\,dx = \frac{1}{2\pi}\int_0^{2\pi} V(-x) f(x)\,dx =$$

$$= \frac{1}{2\pi}\int_0^{-2\pi} V(x) f(-x)(-1)\,dx = \frac{1}{2\pi}\int_{-2\pi}^{0} V(x) f(-x)\,dx = \frac{1}{2\pi}\int_0^{2\pi} V(x) f(-x)\,dx =$$

$$= F_V[\tilde{f}] = \tilde{F}_V(f), \ \forall f \in \mathcal{P}, \text{ so we get}$$

$$F_{\tilde{V}} = \tilde{F}_V, \ \forall V \in \mathcal{E}.$$

3. <u>Translation</u> : given $F \in \mathcal{P}'$, and $t \in \mathbb{R}$, define

$$\left\langle\; T_t F(f) = F(T_{-t} f), \; \forall f \in \mathcal{P} \;\right\rangle$$

Remark : $F_{T_t v}(f) = \frac{1}{2\pi}\int_0^{2\pi} T_t v(x) f(x)\,dx = \frac{1}{2\pi}\int_0^{2\pi} v(x-t) f(x)\,dx =$

$(x-t=y \Rightarrow dx=dy) \;=\; \frac{1}{2\pi}\int_{-t}^{2\pi-t} v(y) f(t+y)\,dy = \frac{1}{2\pi}\int_0^{2\pi} v(y) T_{-t} f(y)\,dy =$

$= F_v[T_{-t} f] = T_t F_v(f)$, $\forall f \in \mathcal{P}$. Then

$$F_{T_t v} = T_t F_v \quad , \forall t \in \mathbb{R}, \forall v \in \mathcal{C}$$

4. <u>Derivatives</u> : given $F \in \mathcal{P}'$, define

$$DF(f) := -F(Df), \; \forall f \in \mathcal{P}$$

and then, inductively,

$$(D^k F)(f) := (-1)^k F(D^k f), \; \forall f \in \mathcal{P}.$$

Remark : if $v \in \mathcal{C}$ happens to be differentiable, integration by parts again give the coincidence of this definition and usual differentiation. The same is true for $v \in \mathcal{C}$ with derivatives of second order, and so on. We have agreement even for $v \in \mathcal{P}$.

5. <u>Some examples</u>

1. Consider $x_0 \in \mathbb{R}$ and $\delta_{x_0}$. Then

$$\delta^*_{x_0}(f) = [\delta_{x_0}(f^*)]^* = [f^*(x_0)]^* = f(x_0) = \delta_{x_0}(f), \; \forall f \in \mathcal{P}$$

We say that a periodic distribution $F \in \mathcal{P}'$ is <u>real</u> if $F = F^*$

We can define $\operatorname{Re} F := \frac{1}{2}(F + F^*)$ and $\operatorname{Im} F := \frac{1}{2i}(F - F^*)$.

It follows that $\delta_{x_0}$ is real.

$$\tilde{\delta}_{x_0}(f) = \delta_{x_0}(\tilde{f}) = \tilde{f}(x_0) = f(-x_0) = \delta_{-x_0}(f) \text{ , so } \tilde{\delta}_{x_0} = \delta_{-x_0}$$

$$T_t \delta_{x_0}(f) = \delta_{x_0}(T_{-t} f) = (T_{-t} f)(x_0) = f(x_0 + t) = \delta_{x_0+t}(f)$$

so we get $\quad T_t \delta_{x_0} = \delta_{x_0+t}$

$$D\delta_{x_0}(f) = -\delta_{x_0}(Df) = -Df(x_0)$$

$$\underline{\underline{D^2 \delta_{x_0}(f) = \delta_{x_0}(D^2 f) = D^2 f(x_0)}}$$

$$\underline{\underline{D^k \delta_{x_0}(f) = (-1)^k \delta_{x_0}(D^k f) = (-1)^k D^k f(x_0).}}$$

Remark: on $\mathcal{P}'$, each of the operations defined above is continuous.
This means, for example, that

$$F_n \xrightarrow{\mathcal{P}'} F \implies F_n^* \xrightarrow{\mathcal{P}'} F^*$$

and so on.

5 **Proposition** (Difference quotients)
Let $F \in \mathcal{P}'$. Then $\left(T_{-t}F - F\right)\frac{1}{t} \xrightarrow[t\downarrow 0]{\mathcal{P}'} DF$

Proof: Fix $f \in \mathcal{P}$.

$$\frac{1}{t}\left(T_{-t}F - F\right)(f) = \frac{1}{t}\left[F(T_t f) - F(f)\right] = F\left[\frac{1}{t}(T_t f - f)\right]$$

Now, remember that $\frac{1}{t}(T_t f - f) \xrightarrow[t\downarrow 0]{\mathcal{P}} -Df$

As $F \in \mathcal{P}'$, we get

$$F\left[\frac{1}{t}(T_t f - f)\right] \longrightarrow F(-Df) = DF(f)$$

## V.7 Description of all periodic distributions

$\mathcal{P}'$ has a subspace consisting of functions (remember that a function is identified with a particular type of distribution, namely given $v \in \mathcal{C}$, define $F_v \in \mathcal{P}'$ by

$$F_v(f) = \frac{1}{2\pi}\int_0^{2\pi} f(t)\,v(t)\,dt \quad , \forall f \in \mathcal{P}$$

We proved earlier that not all elements in $\mathcal{P}'$ are of this kind. In fact, $\delta_{x_0}$ is not $F_v$ for any $v \in \mathcal{C}$, $x_0 \in \mathbb{R}$.
However, the following result says that all periodic distributions can be obtained just adding to functions derivatives of functions.
Precisely

1. **Theorem**: Let $F \in \mathcal{P}'$. Then we can find $v \in \mathcal{C}$, $k \in \mathbb{N} \cup \{0\}$ and a constant function $W$ on $\mathbb{R}$ such that

$$F = D^k F_v + F_W$$

To prove this theorem we need the concept of order of a distribution.
Recall the definition of convergence in $\mathcal{P}$: we use a sequence $S_0, S_1, \dots$ of seminorms instead of a single norm; they are defined

$$S_k(f) = \|D^k f\|, \quad f \in \mathcal{P},\ k \in \mathbb{N} \cup \{0\}$$

2. **Definition**: Let $F \in \mathcal{P}'$. We say that $F$ has order $k$ ($k \in \mathbb{N} \cup \{0\}$) if $\exists c > 0 : |F(f)| \leq c\{\|f\|_\infty + \|Df\|_\infty + \dots + \|D^k f\|_\infty\}$, $\forall f \in \mathcal{P}$

3. **Proposition**: every $F \in \mathcal{P}'$ is of order $k$ for some $k \in \mathbb{N} \cup \{0\}$

Proof: Remember that convergence in $\mathcal{P}$ was given by a distance:

$$d(f,g) = \sum_{k=0}^{\infty} \frac{\|D^k f - D^k g\|_\infty}{1 + \|D^k f - D^k g\|_\infty} \cdot \frac{1}{2^{k+1}}$$

Use now the definition of continuity of $F$ (at $0$):

$$\forall \varepsilon > 0\ \exists \delta > 0 \text{ such that } d(f,0) \leq \delta \Rightarrow |F(f)| \leq \varepsilon$$

It is enough to consider $\varepsilon = 1$. Find $\delta > 0$ accordingly. So we get

$$d(f,0) \leq \delta \Rightarrow |F(f)| \leq 1.$$

$\sum_{0}^{\infty} \frac{1}{2^{k+1}}$ converges, so find $k_0 \in \mathbb{N}$ such that $\sum_{k_0+1}^{\infty} \frac{1}{2^{k+1}} \leq \delta/2$

Given $f \in \mathcal{P}$, compute $\|f\|_\infty + \|Df\|_\infty + \dots + \|D^{k_0} f\|_\infty = \Delta$

Define $g := \frac{\delta}{2\Delta} \cdot f \in \mathcal{P}$. Then

$$\|g\|_\infty + \|Dg\|_\infty + \dots + \|D^{k_0} g\|_\infty = \delta/2$$

It follows that

$$d(g,0) \leq \delta$$

Then $$|F(g)| \leq 1.$$

We get $$|F(f)| = \frac{2\Delta}{\delta} |F(g)| \leq \frac{2}{\delta} \cdot \Delta = \frac{2}{\delta} \left\{ \|f\|_\infty + \|Df\|_\infty + \dots + \|D^{k_0} f\|_\infty \right\}$$

Observe now that $k_0$ does not depend on $f$. We get the conclusion. $\square$

We need also to be able to extend linear and continuous functions from some subspaces to the space itself. There are very powerful extension theorems (like the so called Hahn-Banach Theorem). In our case we need a very simple result:

4. **Lemma**: Consider $F : \mathcal{P} \to \mathbb{C}$ a periodic distribution of order $0$. Then $F$ can be extended to a linear and continuous mapping from $(\mathcal{E}, \|\cdot\|_\infty)$ into $\mathbb{C}$.

Proof: We have $$|F(f)| \leq c \|f\|_\infty, \ \forall f \in \mathcal{P}$$

It follows that $$F : (\mathcal{P}, \|\cdot\|_\infty) \to \mathbb{C} \text{ is continuous.}$$

As $\mathcal{P} \subset (\mathcal{E}, \|\cdot\|_\infty)$ is dense, there is a natural way to define an extension (denoted again by $F$), $F : (\mathcal{E}, \|\cdot\|_\infty) \to \mathbb{C}$:

Given $f \in \mathcal{E}$, there is a sequence $(g_n)$ in $\mathcal{P}$ such that $g_n \xrightarrow{\|\cdot\|_\infty} f$. It follows immediately that $F(g_n) \to$ . This limit is defined as $F(f)$. It is obvious that this definition is independent of the particular sequence $(g_n)$ and that $F$ so defined satisfies all the required properties. $\square$

The order of certain distributions is easily computed:

a) Every function has order 0 :

Let $v \in \mathcal{C}$ and consider $F_v \in \mathcal{P}'$.

$$F_v(f) = \frac{1}{2\pi}\int_0^{2\pi} f(t)v(t)dt \quad , \forall f \in \mathcal{P}$$

$$|F_v(f)| \leq \frac{1}{2\pi}\|f\|_\infty \int_0^{2\pi} |v(t)|dt = c.\|f\|_\infty , \quad \forall f \in \mathcal{P}.$$

Then $F_v$ has order 0.

b) The Dirac delta has order 0 :

Given $x_0 \in \mathbb{R}$, $\delta x_0(f) = f(x_0)$, $\forall f \in \mathcal{P}$. Then

$$|\delta x_0(f)| = |f(x_0)| \leq \|f\|_\infty \quad , \forall f \in \mathcal{P}$$

It follows that $\delta_{x_0}$ has order 0.

c) If $F$ has order $k$, then $DF$ has order $(k+1)$.

Proof: Assume $|F(f)| \leq c\{\|f\|_\infty + \dots + \|D^k f\|_\infty\}$, $\forall f \in \mathcal{P}$.

Then $DF(f) = -F(Df)$, hence

$$|DF(f)| = |F(Df)| \leq c\{\|Df\|_\infty + \dots + \|D^{k+1}f\|_\infty\}, \ \forall f \in \mathcal{P}$$

and $DF$ has order $(k+1)$. □

From those remarks it follows, then, that $D\delta x_0$ is a distribution of order 1, and that $DF_v$ is also of order 1, $\forall v \in \mathcal{C}$.

Remark : There is a certain ambiguity in the definition of order: Assume $|F(f)| \leq c\{\|f\|_\infty + \dots + \|D^k f\|_\infty\}$, $\forall f \in \mathcal{P}$. Obviously $|F(f)| \leq c\{\|f\|_\infty + \dots + \|D^p f\|_\infty\}$, $\forall f \in \mathcal{P}$, $\forall p \geq k$. It follows that if $F$ is of order $k$ it is also of order $k+1, k+2, \dots$ The order should have been defined as the smallest $k$ with the property

$$|F(f)| \leq c\{\|f\|_\infty + \dots + \|D^k f\|_\infty\}, \ \forall f \in \mathcal{P}.$$

Exercise : Prove that if $F$ has order $k$, then $DF$ has order $(k+1)$.

5. $\boxed{F\in\mathcal{P}' \text{ of order } 0 \text{ such that } F(W)=0,\ \forall\, W \text{ constant} \in \mathcal{P}. \text{ Then } F = D^2F_v \text{ for some } v\in\mathcal{C}}$

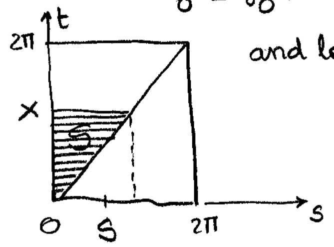
Proof: Assume first $F=F_f$ for some $f\in\mathcal{C}$. We want to prove that $F_f = D^2F_v$ for some $v\in\mathcal{C}$. In fact, this $v$ is going to be two times differentiable, and so $D^2F_v = F_{v''} = F_f$, hence we want $D^2v = f$. Then

$$Dv(x) = a + \int_0^x f(t)\,dt \quad , \quad x\in[0,2\pi]$$

where $a$ is some constant.

Then $$v(x) = ax + \int_0^x \Big[\int_0^t f(s)\,ds\Big]dt$$

Consider




and let $g(s,t) = f(s)$, $(s,t)\in[0,2\pi]\times[0,2\pi]$

$$\int_0^x\Big[\int_0^t f(s)\,ds\Big]dt = \iint_S g(s,t)\,ds\,dt = $$
$$= \int_0^x\Big[\int_s^x g(s,t)\,dt\Big]ds =$$
$$= \int_0^x\Big[f(s)\int_s^x dt\Big]ds = \int_0^x f(s)(x-s)\,ds$$ interchanging the order of integration. We want to write these integrals from 0 to $2\pi$. Then we define $(x-s)^+ = \begin{cases} 0, & \text{if } x-s<0 \\ x-s, & \text{if } x-s\geq 0\end{cases}$. Then

$$v(x) = ax + \int_0^{2\pi} f(s)(x-s)^+\,ds$$

We want a $2\pi$-periodic (continuous) function. $v$ is obviously continuous.

We force $v(2\pi) = v(0) = 0$

$$v(2\pi) = 2\pi a + \int_0^{2\pi} f(s)(2\pi - s)\,ds = 2\pi a - \int_0^{2\pi} s f(s)\,ds = 0$$

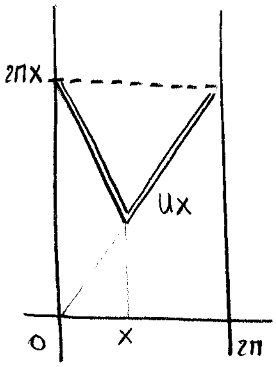
(the last equality because $F_f(W)=0$, $\forall\, W$ constant function).

Hence $$a = \frac{1}{2\pi}\int_0^{2\pi} s f(s)\,ds$$

and then

$$v(x) = \frac{1}{2\pi}\Big[\int_0^{2\pi} xsf(s)\,ds + \int_0^{2\pi} 2\pi\cdot f(s)(x-s)^+\,ds\Big] =$$
$$= \frac{1}{2\pi}\Big[\int_0^{2\pi} f(s)\{xs + 2\pi(x-s)^+\}\,ds\Big] = F_f[u_x]$$

where $\left\langle u_x(s) := xs + 2\pi(x-s)^+,\ s\in[0,2\pi],\ x\in[0,2\pi]\right\rangle$




Then we conclude: assume $F_f \in \mathcal{P}'$ (for some $f \in \mathcal{E}$) such that $F_f(w)=0$, $\forall w \in \mathcal{P}$, $w$ constant function. Then

$$F_f = D^2 F_v$$

for

$$v \in \mathcal{E}, \quad v(x) = F_f[u_x], \quad x \in [0,2\pi]$$

Now we want to deal with an arbitrary $F \in \mathcal{P}'$ such that $F(w)=0$, $\forall w \in \mathcal{P}$, $w$ constant function, and $(F)=0$. We proceed by analogy: Define $\left\langle v \in \mathcal{E},\ v(x) := F[u_x],\ x \in [0,2\pi] \right\rangle$

This definition makes sense in the following sense: $F: \mathcal{P} \to \mathbb{R}$. However $u_x \notin \mathcal{P}$, $u_x \in \mathcal{E}$, whenever $x \in ]0,2\pi[$. As $\operatorname{od}(F)=0$, apply Lemma 4. Then $F$ can be extended to $\mathcal{E}$. This allows us to compute $F[u_x]$.

$$\boxed{\text{We shall prove that } v \in \mathcal{E} \text{ and } D^2 F_v = F}$$

☒ Note that $u_0 \equiv 0$ and $u_{2\pi}(s) = 2\pi s + 2\pi(2\pi - s) = 4\pi^2$, hence $F[u_0] = 0 = F[u_{2\pi}]$ and then $v(0)=v(2\pi)$. We extend $v$ to $\mathbb{R}$ by periodicity. To prove that $v$ is continuous, let $0 \le x < y \le 2\pi$, $s \in [0,2\pi]$.

$$|u_y(s) - u_x(s)| = |ys + 2\pi(y-s)^+ - xs - 2\pi(x-s)^+| = [1]$$

a) $0 \le s \le x \Rightarrow [1] = |(y-x)s + 2\pi(y-s-x+s)| \le 4\pi|y-x|$

b) $y < s \le 2\pi \Rightarrow [1] = |(y-x)s| \le 2\pi|y-x|$

c) $x < s \le y \Rightarrow [1] = |(y-x)s + 2\pi(y-s)| \le 4\pi|y-x|$

Then $|v(y)-v(x)| = |F[u_y] - F[u_x]| \le \|F\| \cdot \|u_y - u_x\| \le 4\pi \|F\| \cdot |y-x|$

and then $v$ is continuous.

To calculate $DF_v$ write, for $w \in \mathcal{P}$, and $t \neq 0$,

$$2\pi \frac{1}{t}\left[T_{-t}F_v - F_v\right](w) = \frac{1}{t}\int_0^{2\pi} [v(x+t) - v(x)]\, w(x)\, dx$$

Approximate this integral by its Riemann sums (take a partition of $[0,2\pi]$):

$$\frac{1}{t}\sum [v(x_i+t) - v(x_i)]\, w(x_i)\,(x_i - x_{i-1}) =$$

$$= \frac{1}{t}\sum [F(u_{x_i+t}) - F(u_{x_i})]\, w(x_i)\,(x_i - x_{i-1}) =$$

$$= F\left[\sum \frac{1}{t} w(x_i)\{u_{x_i+t} - u_{x_i}\}(x_i - x_{i-1})\right]$$

Facts on functions which approach (in $\|\cdot\|_\infty$) to the function $g_t$, where

$$g_t(s) := \int_0^{2\pi} \frac{1}{t} W(x)\,[u_{x+t} - u_x](s)\,dx$$

Those are integrals depending on a parameter $t$. Fix $s \in [0, 2\pi]$

If $0<x<s$, $\frac{u_{x+t} - u_x}{t}(s) = \frac{1}{t}\left[(x+t)s + 2\pi (x+t-s)^+ - xs - 2\pi (x-s)\right] =$

$$= \frac{1}{t}[ts] = s \longrightarrow s \quad \text{when } t \to 0.$$

If $s<x<2\pi$ $\frac{u_{x+t} - u_x}{t}(s) = \frac{1}{t}\left[(x+t)s + 2\pi(x+t-s) - xs - 2\pi(x-s)\right] =$

$$= \frac{1}{t}\left[ts + 2\pi t\right] = s + 2\pi \longrightarrow s+2\pi \quad \text{when } t \to 0$$

Recall that

$$2\pi \frac{1}{t}\left[T_{-t}F_v - F_v\right] \xrightarrow[t \to 0]{} (DF_v) \cdot 2\pi \qquad (\text{in the } w^*\text{- topology})$$

Then $\qquad 2\pi(DF_v)(W) = F(g)$

$$\text{where } g(s) = \int_0^s s\,W(x)\,dx + \int_s^{2\pi} W(x)\,[s+2\pi]\,dx =$$

$$= s\int_0^{2\pi} W(x)\,dx + \int_s^{2\pi} 2\pi \cdot W(x)\,dx$$

We get

$2\pi\left[D^2 F_v\right](w) = -2\pi\, DF_v[DW] = -F(h)$, where

$$h(s) = s\int_0^{2\pi} DW(x)\,dx + 2\pi \int_s^{2\pi} DW(x)\,dx =$$

$$= s\left[W(2\pi) - W(0)\right] + 2\pi\left[W(2\pi) - W(s)\right]$$

Then $\qquad 2\pi D^2F_v(W) = 2\pi F[W]$, as $F(\text{cte}) = 0$.

We get $\boxed{D^2F_v = F}$ $\square$

6. Proposition

Let $F \in \mathcal{P}'$, $F(w)=0$, $\forall w \in \mathcal{P}$, $w$ constant. Then there is a unique $G \in \mathcal{P}'$ such that $DG = F$, $G(w)=0$, $\forall w$ constant. Moreover, if $\mathrm{ord}(F) = k \geq 1$ then $\mathrm{ord}(G) = k-1$.

Proof: Define $\quad Su(x) = \int_0^x u(t)dt - \frac{x}{2\pi}\int_0^{2\pi} u(t)dt = \int_0^x u(t)dt - x F_e(u)$

where $e$ is the constant function 1.

Then $\quad D(Su) = u - F_e(u)e$

Assume $\quad DG = F$. Then $\ G(u) = G(u - F_e(u)e)$, as $G(e) = 0$.

Then $\quad G(u) = G(D(Su)) = -DG(Su) = -F[Su]$, so $G$ is uniquely defined.

In order to prove that $G$ exists, define

$$G(u) = -F(Su).$$

We shall prove that $G \in \mathcal{P}'$:

First, observe that $S : \mathcal{P} \to \mathcal{P}$ is linear and continuous:

Linearity is obvious. To check continuity is enough to do it at 0:

$\|Su\|_\infty \leq 2\pi\|u\|_\infty + 2\pi\|F_e u\|_\infty \leq 2\pi\|u\|_\infty + 2\pi\|u\|_\infty = 4\pi\|u\|_\infty$

$\|DSu\|_\infty \leq \|u\|_\infty + |F_e(u)|\cdot\|e\|_\infty \leq 2\|u\|_\infty$

$\|D^k Su\|_\infty \leq \|D^{k-1}u\|_\infty, \quad k \geq 2.$

This proves $S : \mathcal{P} \to \mathcal{P}$ continuous at 0 (obviously, $Su \in \mathcal{P}$, $\forall u \in \mathcal{P}$)

It follows that $G$ is continuous: $\mathcal{P} \to \mathbb{C}$.

Moreover $\quad DG(u) = -G(Du) = F(SDu) = F(u)$, $\forall u \in \mathcal{P}$

hence $DG = F$.

Finally $\quad |G(u)| = |F(Su)| \leq C\{\|Su\|_\infty + \|DSu\|_\infty + \dots + \|D^k Su\|_\infty\} \leq$

$$\leq C\{4\pi\|u\|_\infty + 2\|u\|_\infty + \|Du\|_\infty + \dots + \|D^{k-1}u\|_\infty\}$$

so $G$ has order $(k-1)$. $\square$

The following corollary determines distributions with zero derivative:

7. Corollary: A distribution $G \in \mathcal{P}'$ satisfies $DG \equiv 0$ if and only if $G = F_f$ for some constant function $f \in \mathcal{E}$.

Proof: Assume $f \in \mathcal{E}$ is constant. Then $DF_f(u) = -F_f(Du) =$

$$= \frac{-1}{2\pi}\int_0^{2\pi} D(u)(t)\,dt \cdot f(t) = \frac{-1}{2\pi} f \int_0^{2\pi} Du(t)\,dt = \frac{-1}{2\pi} f \cdot [u(2\pi) - u(0)] = 0$$

On the other hand, let $f \equiv G(e).e$, where $e$ is the constant 1 function. Let

$$H := G - F_f$$

Then $DH = DG - DF_f = DG = 0$ and $H(e) = G(e) - F_f(e) =$

$$= G(e) - \frac{1}{2\pi}\int_0^{2\pi} f(t)\,dt = G(e) - G(e) = 0$$

It follows that $H$ satisfies $DH = 0$, $H(w) = 0$, $\forall\, w$ constant function, the same properties of $0$. By uniqueness, $H \equiv 0$.

Proof of Theorem 1

$F \in \mathcal{P}'$. Let $k \in \mathbb{N}$ such that $F$ is of order $K \geq 2$ (may be $K$ is not the minimum). Let $e \equiv 1$ and $f \equiv F(e).e$. Let

$$F_0 \equiv F - F_f$$

Then $F_0$ is of order $K$ and $F_0(e) = F(e) - \frac{1}{2\pi}\int_0^{2\pi} f(t)\,e(t)\,dt =$

$$= F(e) - \frac{1}{2\pi} F(e)\int_0^{2\pi} dt = 0.$$

Using the lemma, $\exists\, F_{K-1}$ such that $DF_{K-1} = F_0$ and $F_{K-1}(w) = 0$, $\forall w$ constant function. Proceed recursively to find $F_{K-2}, \dots, F_0$.

As $F_0$ has order $0$ and $F_0(e) = 0$, then $F_0 = D^2 F_v$ for some $v \in \mathcal{E}$.

$$F_0 = DF_{K-1} = D^2 F_{K-2} = \dots = D^K F_0 = D^{K+2} F_v$$

$$F = F_0 + F_f = D^{K+2} F_v + F_f. \quad \square$$

Remark on Riemann sums

Let $h : [a, b] \to \mathbb{R}$ be a continuous function on $[a, b]$. Then $h$ is uniformly continuous; it follows that given $\varepsilon > 0$, there exists $\delta > 0$ such that

$$|x - y| < \delta \text{ in } [a b] \to |h(x) - h(y)| < \varepsilon$$

Now, assume that $a = x_0 < x_1 < \dots < x_n = b$ is a partition of $[a b]$ such that $|x_i - x_{i-1}| < \delta$, $i = 1, 2, \dots, n$. Then, given $i \in \{1, 2, \dots, n\}$

$$\int_{x_{i-1}}^{x_i} h(x)\,dx \in [m_i (x_i - x_{i-1}), M_i (x_i - x_{i-1})]$$

where $m_i = \min_{x \in [x_{i-1}, x_i]} h(x)$, $M_i = \max_{x \in [x_{i-1}, x_i]} h(x)$, and $M_i - m_i \leqslant \varepsilon$

It follows that

$$\left| \int_a^b h(x)\,dx - \sum_{i=1}^n h(x_i)(x_i - x_{i-1}) \right| = \left| \sum_{i=1}^n \left[ \int_{x_{i-1}}^{x_i} h(x)\,dx - h(x_i)(x_i - x_{i-1}) \right] \right|$$

$$\leqslant \sum_{i=1}^n \varepsilon (x_i - x_{i-1}) = \varepsilon (b - a)$$

Now consider $g_t(s) := \int_0^{2\pi} \frac{1}{t} W(x) [u_{x+t} - u_x](s)\,dx$. Fix $t$ and $s$. We want to approximate this integral by a Riemann sum. Fix $\varepsilon > 0$. The possibility to control the approximation related to the mess of the partition depends on the uniform continuity of the three variables function

$$(t, s, x) \to \frac{1}{t} W(x) [u_{x+t}(s) - u_x(s)]$$

This is a continuous function on the compact set $[-1, 1] \times [0, 2\pi] \times [0, 2\pi]$ and then we can find $\delta > 0$ such that if

$$0 = x_0 < x_1 < \dots < x_n = 2\pi$$

has a mess $< \delta$ then

$$\left| g_t(s) - \sum \frac{1}{t} W(x_i) [u_{x_i+t}(s) - u_{x_i}(s)] (x_i - x_{i-1}) \right| < \varepsilon \qquad [*]$$

whenever $t \in [-1, 1]$, $s \in [0, 2\pi]$

If we consider both summands in $[*]$ as functions of $s$, we get

$$\left\| g_t - \sum \frac{1}{t} W(x_i) [u_{x_i+t} - u_{x_i}] (x_i - x_{i-1}) \right\|_\infty \leqslant \varepsilon$$

## V.8 Some remarks on order

1 Let $F \in \mathcal{P}'$ a periodic distribution. Define

F has <u>order k</u> ($k \in \mathbb{N} \cup \{0\}$) if there exists $c > 0$ such that

$$|F(u)| \leq c\{\|u\|_\infty + \|Du\|_\infty + \dots + \|D^k u\|_\infty\}, \ \forall u \in \mathcal{P}$$

<u>Remark</u>: As soon as we can say that F has order k for some $k \in \mathbb{N} \cup \{0\}$ it is obviously true that F has also order m for $m \geq k$. Then it makes sense to define <u>the order</u> of a distribution as the smallest $k \in \mathbb{N} \cup \{0\}$ such that F has order k.

<u>Some examples</u>:

① Given $x_0 \in [0, 2\pi]$, $\delta_{x_0}$ has order 0 (and then the order is 0)

<u>Proof</u>: $|\delta_{x_0}(u)| = |u(x_0)| \leq \|u\|_\infty$, $\forall u \in \mathcal{P}$. Then $\delta_{x_0}$ has order 0. □

② Given $f \in \mathcal{C}$, $F_f$ has order 0 (and then the order is 0)

<u>Proof</u> $|F_f(u)| = \frac{1}{2\pi}\left|\int_0^{2\pi} f(x)u(x)dx\right| \leq \frac{1}{2\pi}\|u\|_\infty \int_0^{2\pi}|f(x)|dx =$

$= \frac{c}{2\pi}\|u\|_\infty$, $\forall u \in \mathcal{P}$, where $c = \int_0^{2\pi}|f(x)|dx$. □

2 <u>Lemma</u>: Assume $F \in \mathcal{P}'$ has order $k \in \mathbb{N} \cup \{0\}$. Then DF has order $(k+1)$

<u>Proof</u>: $|F(u)| \leq c\{\|u\| + \dots + \|D^k u\|_\infty\}$, $\forall u \in \mathcal{P}$.

Then $|DF(u)| = |F(Du)| \leq c\{\|Du\| + \dots + \|D^{k+1}u\|_\infty\}$, $\forall u \in \mathcal{P}$

so DF has order $(k+1)$. □

<u>Remark</u>: It is tempting to say that if F is of order k then DF is of order $(k+1)$. This is true in some situations, but not always: To see that, consider first $f \in \mathcal{C}$ such that f is differentiable (at least once). Then

$$DF_f(u) = -F_f(Du) = -\frac{1}{2\pi}\int_0^{2\pi} f(t)Du(t)dt = -\frac{1}{2\pi}\left\{f(t)u(t)\Big|_0^{2\pi} - \int_0^{2\pi} Df(t)u(t)dt\right\} =$$

$$= +\frac{1}{2\pi}\int_0^{2\pi} Df(t)u(t)dt = F_{Df}(u)$$ , $\forall u \in \mathcal{P}$, hence

$$DF_f = F_{Df}$$

It follows that the particular distribution $F_f$ ( the order of this distribution is $0$ ) has a derivative $DF_f = F_{Df}$ (and the order of this distribution is again $0$).

In case of $F = \delta_{x_0}$ (a distribution whose order is $0$) the derivative has order $1$ (in fact, is of order $1$)

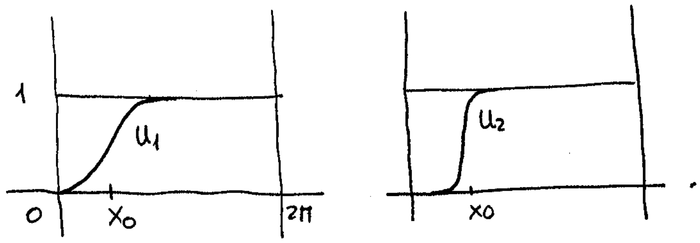
Proof: $D\delta_{x_0}(u) = -\delta_{x_0}(Du) = -Du(x_0)$

We know, by the lemma, that $D\delta_{x_0}$ has order $1$. Let us prove that it has not order $0$. Assume for a moment this is the case:

$$|D\delta_{x_0}(u)| \leq c\|u\|_\infty \quad \text{for } u \in \mathcal{P} \text{ and some } c>0, \text{ i.e.}$$

$$|Du(x_0)| \leq c\|u\|_\infty, \ \forall u \in \mathcal{P}$$

It is enough, to reach a contradiction, to build a sequence $(u_n)_1^\infty$ in $\mathcal{P}$ such that $Du_n(x_0) \longrightarrow +\infty$ and $\|u_n\|_\infty \leq 1$. This is simple:



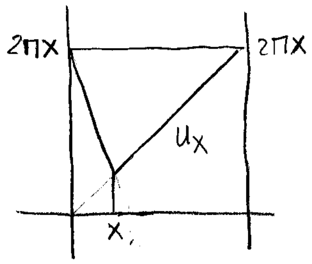

Exercise: Prove that the order of $D^k\delta_{x_0}$ is precisely $k$.

Exercise: Given $x_0 \in [0, 2\pi]$, find $v \in \mathcal{E}$ and $f \in \mathcal{E}$, $f$ a constant function, such that $\delta_{x_0} = D^2F_v + F_f$.

3. In particular, every Dirac delta can be obtained in this way: there exists a continuous periodic function $V$ and a constant function $f$ such that $\delta_{x_0} = D^k F_V + F_f$, for some $k \in \{0,1,2,..\}$.
In order to find $V$, $f$ and $k$ we can proceed in the following way:

1) Given $e \in \mathcal{P}$, $e$ a constant function identically $1$, we have
$$\delta_{x_0}(e) = 1 = \frac{1}{2\pi}(-1)^k \int_0^{2\pi} V(x)\, D^k e(x)\,dx + \frac{1}{2\pi}\int_0^{2\pi} M\,dx = M$$
where $f(x) = M$, $\forall x$. Then $f \equiv e$
$$\delta_{x_0} = D^k F_V + F_e$$

2) Consider now $\delta_{x_0} - F_e$, an element in $\mathcal{P}'$.
We know that $G := \delta_{x_0} - F_e$ has order $0$ (in fact, the sum of two distributions of order $k$ is also of order $k$)
Let $u \in \mathcal{P}$ be a constant function. Then, if $u(x) = M$, $\forall x$,
$$(\delta_{x_0} - F_e)\, u = u(x_0) - \frac{1}{2\pi}\int_0^{2\pi} M\,dx = 0$$
So we are in the situation of Lemma V.7.5
There exists a function $V \in \mathcal{E}$ such that $\delta_{x_0} - F_e = D^2 F_V$. This Lemma gives the precise description of $V$:
Define

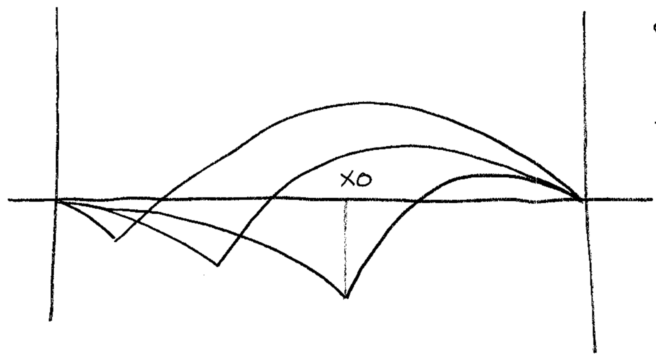



$$U_X(s) = Xs + 2\pi(X-s)^+, \quad s \in [0, 2\pi]$$
Then, consider the canonical extension of $G$ to $\mathcal{E}$: precisely
$$G(u) = \delta_{x_0}(u) - F_e(u) = u(x_0) - \frac{1}{2\pi}\int_0^{2\pi} u(x)\,dx, \quad \forall u \in \mathcal{P}$$
Then $\quad G(h) = h(x_0) - \frac{1}{2\pi}\int_0^{2\pi} h(x)\,dx, \ \forall h \in \mathcal{E}$

Define
$$V(x) = G(U_x) = U_x(x_0) - \frac{1}{2\pi}\int_0^{2\pi} U_x(s)\,ds = U_x(x_0) - \left(\pi x + \frac{x^2}{2}\right)$$

a) $x \in [0, x_0] \Rightarrow V(x) = x x_0 - \pi x - \frac{x^2}{2} = \frac{-x^2}{2} + x(x_0 - \pi)$

b) $x \in [x_0, 2\pi] \Rightarrow V(x) = x x_0 + 2\pi(x - x_0) - \pi x - \frac{x^2}{2} = -\frac{x^2}{2} + x(x_0 + \pi) - 2\pi x_0$




Several graphs corresponding to different $\delta_{x_0}$

Exercises

① The following expressions define functions on $\mathcal{P}$. Determine which of them are elements in $\mathcal{P}'$ (here, $u \in \mathcal{P}$):

a) $F(u) = Du(1) - 3u(2\pi)$

b) $F(u) = \int_0^{2\pi} [u(x)]^2 dx$

c) $F(u) = \int_0^{2\pi} u(x)\, dx$

d) $F(u) = \int_0^{\pi} u(x)[(1+x)]^7 dx$

e) $F(u) = -\int_0^{2\pi} D^3 u(x)\, |\cos 2x|\, dx$

f) $F(u) = \sum_{j=0}^{n} a_j D^j u(t_j)$

g) $F(u) = \sum_{j=0}^{\infty} (j!)^{-1} D^j u(0)$

② a) and f) are actually distributions, and they can be expressed by using $\delta$'s. Do it.

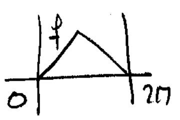
③ c) and d) are actually distributions. Compute $DF$

④ Let $v(x) = \left|\sin \frac{x}{2}\right|$. Compute $D^2 F_v$

⑤ A derivation is defined on distributions. Prove that this operation is an extension of the classical one: precisely; assume that $v \in \mathcal{E}$ can be differentiated. Prove that

$$DF_v = F_{Dv}$$

⑥ In the notes, it is proved that $\delta_{x_0} = D^2 F_V + F_W$ for some $V \in \mathcal{E}$ and some constant function $W \in \mathcal{E}$. Observe then the aspect of $D^k \delta_{x_0}$ in terms of $F_V$ and $F_W$.

⑦ Consider the function 

Identify $F_f$, $DF_f$, ....

## V.9 $\boxed{\text{Convolution of distributions}}$

1\. The purpose is to define $F*G$ as an element in $\mathcal{P}'$, given $F, G \in \mathcal{P}'$.
As in previous situations, we want to do it keeping compatibility with the definition of $f*g$ for $f, g \in \mathcal{E}$. Precisely we want

$$F_f * F_g = F_{f*g} \quad , \forall f, g \in \mathcal{E}.$$

The definition is done in two steps

(I) Let $F \in \mathcal{P}'$, $u \in \mathcal{P}$. Define

$$(F*u)(x) = F(T_x \tilde{u}), \forall x \in \mathbb{R} \qquad [*]$$

We shall prove that $F*u \in \mathcal{P}$

(II) Let $F \in \mathcal{P}'$, $G \in \mathcal{P}'$. We define

$$(F*G)(u) = F[\tilde{G}*u] \quad , \forall u \in \mathcal{P} \qquad [**]$$

We shall prove that $F*G \in \mathcal{P}'$

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2\. (I) Simple properties of the operation defined in $[*]$ are

1) $(\alpha F)*u = \alpha(F*u) = F*(\alpha u)$, $\forall \alpha \in \mathbb{C}$
2) $(F+G)*u = (F*u) + (G*u)$
3) $F*(u+v) = (F*u) + (F*v)$
4) $T_t(F*u) = (T_t F)*u = F*(T_t u)$, $\forall t \in \mathbb{R}$
5) $D(F*u) = (DF)*u = F*(Du)$

The proof is easy. For example, in order to get (5),

$$\frac{1}{t}\left[(F*u)(x+t) - (F*u)(x)\right] = \frac{1}{t}\left[T_{-t}(F*u) - F*u\right](x) =$$

$$= \frac{1}{t}\left[(T_{-t}F)*u - F*u\right](x) = \left[\frac{1}{t}\left[T_{-t}F - F\right]*u\right](x) =$$

$$= \frac{1}{t}(T_{-t}F - F)\left[T_x\tilde{u}\right] \xrightarrow{t\to 0} DF\left[T_x\tilde{u}\right] = (DF*u)(x)$$

It follows that $(F*u)$ has a derivative, and

$$D(F*u) = (DF)*u$$

By using the second part of (4) instead of the first part we get

$$D(F*u) = F*(Du).$$

The k'th derivative is obtained by induction. □
This proves $(F*u) \in \mathcal{P}$.

3. (II) By (I) the operation introduced in [**] is well defined.

$$(F*G)(u+v) = F[\tilde{G}*(u+v)] = F[\tilde{G}*u + \tilde{G}*v] = F[\tilde{G}*u] + F[\tilde{G}*v]$$
$$= (F*G)(u) + (F*G)(v)$$

Moreover $(F*G)(\lambda u) = \lambda\,(F*G)(u)$, $\forall \lambda \in \mathbb{C}$, hence

$F*G$ is a linear function on $\mathcal{P}$

We need to prove that it is continuous on $(\mathcal{P}, d)$.

Assume $u_k \xrightarrow{\mathcal{P}} u$ in $\mathcal{P}$.

$$(F*G)(u_k) = F[\tilde{G}*u_k]$$

We need to prove that $\tilde{G}*u_k \xrightarrow{\mathcal{P}} \tilde{G}*u$

$$|\tilde{G}*u_k(x) - \tilde{G}*u(x)| = |\tilde{G}[T_x\tilde{u}_k] - \tilde{G}[T_x\tilde{u}]| =$$
$$= |\tilde{G}[T_x\tilde{u}_k - T_x\tilde{u}]|$$

Now, $T_x\tilde{u}_k \xrightarrow{\|\cdot\|_\infty} T_x\tilde{u}$ uniformly on $x$ and we get then

$$\tilde{G}*u_k \xrightarrow{\|\cdot\|_\infty} \tilde{G}*u$$

Analogously for the derivatives. $\square$

4. <u>Compatibility of the definition with the classical convolution:</u>

Assume $f, g \in \mathcal{E}$. We shall prove

$$F_{f*g} = F_f * F_g$$

$$F_{f*g}(u) = \frac{1}{2\pi}\int_0^{2\pi} (f*g)(t)\,u(t)\,dt = \frac{1}{2\pi}\int_0^{2\pi}\left[\frac{1}{2\pi}\int_0^{2\pi} f(s)\,g(t-s)\,ds\right]u(t)\,dt$$

$$(F_f * F_g)(u) = F_f[\tilde{F}_g * u] = \frac{1}{2\pi}\int_0^{2\pi} f(s)\,[\tilde{F}_g * u](s)\,ds =$$
$$= \frac{1}{2\pi}\int_0^{2\pi} f(s)\,[\tilde{F}_g(T_s\tilde{u})]\,ds = \frac{1}{2\pi}\int_0^{2\pi} f(s)\,F_g(T_s\tilde{u})\,ds =$$
$$= \frac{1}{2\pi}\int_0^{2\pi} f(s)\,\frac{1}{2\pi}\int_0^{2\pi} \tilde{g}(t)\,T_s\tilde{u}(t)\,dt\,ds = \left(\frac{1}{2\pi}\right)^2 \int_0^{2\pi} f(s)\,g(-t)\,\tilde{u}(t-s)\,dt\,ds =$$
$$= \left(\frac{1}{2\pi}\right)^2 \int_0^{2\pi} f(s)\,g(-t)\,u(s-t)\,dt\,ds = \left(\frac{1}{2\pi}\right)^2 \int_0^{2\pi} f(s)\left(\int_0^{2\pi} g(-t)\,u(s-t)\,dt\right)ds =$$
$$= \left(\frac{1}{2\pi}\right)^2 \int_0^{2\pi} f(s)\left(\int_0^{2\pi} g(t-s)\,u(t)\,dt\right)ds = F_{f*g}(u). \quad \square$$

## 5. Another question of compatibility

$u \in \mathcal{P}$, $F \in \mathcal{P}'$. We have now two different paths

$$(F, u) \longrightarrow F*u \in \mathcal{P} \longrightarrow F_{F*u} \in \mathcal{P}'$$
$$(F, u) \longrightarrow F*F_u \in \mathcal{P}'$$

$$F_{F*u}(v) = \frac{1}{2\pi}\int_0^{2\pi} (F*u)(t)\, v(t)\, dt = \frac{1}{2\pi}\int_0^{2\pi} F[T_t \tilde{u}] \cdot v(t)\, dt$$

$$(F*F_u)(v) = F\,(\tilde{F}_u * v)$$

$$\sum F[T_{t_i}\tilde{u}]\, v(t_i)(t_i - t_{i-1}) = F\left[\sum (T_{t_i}\tilde{u})\, v(t_i)(t_i - t_{i-1})\right]$$

The function on which $F$ acts, as the mess of the partition goes to zero, approaches (uniformly)

$$\int_0^{2\pi} T_t \tilde{u}(s)\, v(t)\, dt = g(s) = \int_0^{2\pi} u(t-s)\, v(t)\, dt$$

$$(\tilde{F}_u * v)(s) = \tilde{F}_u[T_s \tilde{v}] = F_{\tilde{u}}[T_s\tilde{v}] =$$
$$= \frac{1}{2\pi}\int_0^{2\pi} \tilde{u}(t)(T_s\tilde{v})(t)\, dt = \frac{1}{2\pi}\int_0^{2\pi} u(-t)\, v(s-t)\, dt = (s-t=y)$$
$$= \frac{1}{2\pi}\int_s^{s-2\pi} u(y-s)\, v(y)\,(-dy) = \frac{1}{2\pi}\int_0^{2\pi} u(t-s)\, v(t)\, dt$$

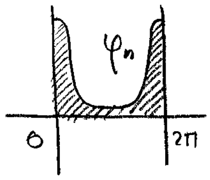
and then

$$\boxed{F * F_u = F_{F*u}, \quad \forall F \in \mathcal{P}', \forall u \in \mathcal{P}}$$

## 6. Convolution and approximation

Recall that an approximate identity $(\varphi_n)$ is a sequence of elements in $\mathcal{C}$ such that

1. $\varphi_n \geq 0$, $\forall n$
2. $\frac{1}{2\pi}\int_0^{2\pi} \varphi_n(t)\,dt = 1$, $\forall n$
3. $0<\delta<2\pi-\delta<2\pi \Rightarrow \int_\delta^{2\pi-\delta} \varphi_n(t)\,dt \xrightarrow[n\to\infty]{} 0$




It was proved that, given $u \in \mathcal{C}$, $\varphi_n * u \xrightarrow{\|\cdot\|_\infty} u$
and, given $u \in \mathcal{P}$, $\varphi_n * u \xrightarrow{\mathcal{P}} u$

Now, we have

**Proposition**: Let $(\varphi_n)$ be an approximate identity in $\mathcal{P}$. Then

$$F_{\varphi_n} * F \xrightarrow{\mathcal{P}'} F$$

Proof: given $u \in \mathcal{P}$

$$(F_{\varphi_n} * F)(u) = (F * F_{\varphi_n})(u) = F[\tilde{F}_{\varphi_n} * u] = F[F_{\tilde{\varphi}_n} * u] =$$
$$= F[\tilde{\varphi}_n * u]$$

Now, $\tilde{\varphi}_n * u \xrightarrow{\mathcal{P}} u$, as $(\tilde{\varphi}_n)$ is also an approximate identity. It follows that $F[\tilde{\varphi}_n * u] \longrightarrow F[u]$. $\square$

Remark: define $f_n := F * \varphi_n \in \mathcal{P}$, $\forall n$. Then

$$F_{f_n} = F_{\varphi_n} * F \xrightarrow{\mathcal{P}'} F$$

and $(f_n)$ is a sequence of trigonometric polynomials:
Let $e_k(x) = \exp(2\pi i k x)$; then

$$(F * e_k)(x) = F[T_x \tilde{e}_k]$$
$$T_x \tilde{e}_k(y) = e_k(x-y) = e_k(x) \cdot \tilde{e}_k(y)$$
$$(F * e_k)(x) = e_k(x) \cdot F(\tilde{e}_k)$$

so $F * (\text{trigonometric polynomial}) = \text{trigonometric polynomial}$.

7. <u>Remark</u>

The natural extensions of properties of (I) are also valid for (II)

1) $F * G = G * F$
2) $(\alpha F) * G = \alpha (F * G) = F * (\alpha G)$
3) $(F+G) * H = (F * H) + (G * H)$
4) $(F * G) * H = F * (G * H)$
5) $T_t (F * G) = (T_t F) * G = F * (T_t G)$
6) $D^k (F * G) = (D^k F) * G = F * (D^k G)$

## VI — A Hilbert space of periodic distributions

**6.1** $\mathcal{C}$ denotes the space of continuous $2\pi$-periodic functions from $\mathbb{R}$ into $\mathbb{C}$. We endowed $\mathcal{C}$ with the norm

$$\|f\|_\infty := \sup\{|f(x)| : x \in \mathbb{R}\} = \sup\{|f(x)| : x \in [0, 2\pi]\}$$

Now we introduce another norm in $\mathcal{C}$:

$$\|f\|_2 := \left(\frac{1}{2\pi}\int_0^{2\pi} |f(x)|^2 dx\right)^{1/2}$$

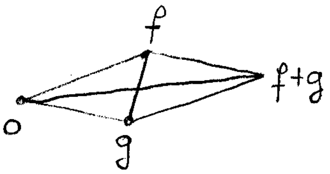
This norm comes from an inner product on $\mathcal{C} \times \mathcal{C}$:

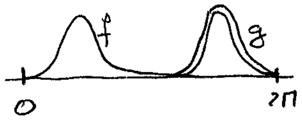
$$\langle f, g \rangle := \frac{1}{2\pi}\int_0^{2\pi} f(x) \cdot g^*(x) dx, \quad f, g \in \mathcal{C},$$

where $*$ denotes complex conjugacy.

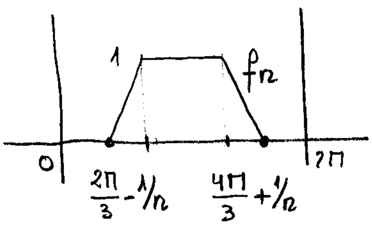
**6.2** Remarks ① $\|\cdot\|_\infty$ does not come from an inner product, as it does not satisfies the paralelogram identity

$$\|f+g\|^2 + \|f-g\|^2 = 2\|f\|^2 + 2\|g\|^2$$




② $(\mathcal{C}, \|\cdot\|_\infty)$ is a complete space. However, $(\mathcal{C}, \|\cdot\|_2)$ is not complete (i.e., it is a normed space but not a Banach space). In order to see that, consider the sequence $(f_n)$:




③ Every normed space can be placed inside a Banach space in which it is dense. This over-space is called the completion and it is, essentially, unique. However, it is, in a sense, only a « theoretical » construction. It is very important, for some particular classes, to have a precise, « practical » description of these completions.

④ $\langle\,\rangle$ is an inner product in $\mathcal{C}$. Then $|\langle f, g\rangle| \leq \|f\|_2 \cdot \|g\|_2$ (Cauchy-Schwarz inequality).

⑤ $\|f\|_2 \leq \|f\|_\infty, \ \forall f \in \mathcal{C}$.

## 6.3 The completion of $(\mathcal{C}, \|\cdot\|_2)$

To get an adequate description of this completion it is better to identify $\mathcal{C}$ with a subspace of $\mathcal{P}'$, as it has been done previously. The identification associates $F_f \in \mathcal{P}'$ to $f \in \mathcal{C}$. Recall that

6.3.1 $$F_f(u) = \frac{1}{2\pi}\int_0^{2\pi} f(t)u(t)\,dt, \ \forall u \in \mathcal{P}$$

**Lemma**: Let $(f_n)_1^\infty$ be a $\|\cdot\|_2$-Cauchy sequence in $\mathcal{C}$. Then there exists a (unique) $F \in \mathcal{P}'$ such that $F_{f_n} \xrightarrow{\mathcal{P}'} F$. Moreover, $F$ is a distribution of order $0$.

Proof: Let $u \in \mathcal{P}$. Then

$$\left|\left(F_{f_n} - F_{f_m}\right)(u)\right| = \left|\frac{1}{2\pi}\int_0^{2\pi} u(x)\left[f_n(x) - f_m(x)\right]dx\right| = \left|\langle \overset{*}{u}, f_n - f_m\rangle\right| \leq$$
$$\leq \|\overset{*}{u}\|_2 \cdot \|f_n - f_m\|_2 \longrightarrow 0 \quad \text{as } n, m \to +\infty.$$

It follows that $\left(F_{f_n}(u)\right)_{n=1}^\infty$ converges (to some limit denoted by $F(u)$), $\forall u \in \mathcal{P}$. Obviously $F : \mathcal{P} \to \mathbb{C}$ is a linear functional. We shall prove that $F \in \mathcal{P}'$ and $F$ is of order $0$.

As $(f_n)_1^\infty$ is $\|\cdot\|_2$-Cauchy, it is a $\|\cdot\|_2$-bounded sequence. Then $\|f_n\|_2 \leq M, \forall n$. Given $u \in \mathcal{P}$,

$$\left|F_{f_n}(u)\right| = \left|\langle u, \overset{*}{f_n}\rangle\right| \leq \|u\|_2 \cdot \|\overset{*}{f_n}\|_2 \leq M \cdot \|u\|_2, \ \forall n \in \mathbb{N}. \qquad [*]$$

6.3.2 Then $|F(u)| \leq M\|u\|_2 \leq M \cdot \|u\|_\infty$, $\forall u \in \mathcal{P}$ and then $F \in \mathcal{P}'$ and has order $0$. □

**Lemma**: Let $(f_n)_1^\infty$ be a $\|\cdot\|_2$-Cauchy sequence in $\mathcal{C}$. Then $F_{f_n} \xrightarrow{\mathcal{P}'} 0$ if and only if $f_n \xrightarrow{\|\cdot\|_2} 0$

Proof: Assume $\|f_n\|_2 \to 0$. Then from $[*]$ above,

$$\left|F_{f_n}(u)\right| \leq \|u\|_2 \cdot \|\overset{*}{f_n}\|_2 \longrightarrow 0 \quad \text{when } n \to \infty, \ \forall u \in \mathcal{P}. \text{ It follows}$$

that $F_{f_n} \xrightarrow{\mathcal{P}'} 0$.

Assume now that $F_{f_n} \xrightarrow{\mathcal{P}'} 0$. Given $\varepsilon > 0$, find $n_0 / \forall n, m \geq n_0$

$$\|f_n - f_m\|_2 < \varepsilon.$$

Now,
$$\|f_n\|_2^2 = \langle f_n, f_n\rangle = \langle f_n, f_n - f_m\rangle + \langle f_n, f_m\rangle =$$
$$= \langle f_n, f_n - f_m\rangle + F_{f_m}(\overset{*}{f_n}) \leq \|f_n\|_2 \cdot \|f_n - f_m\|_2 + \left|F_{f_m}(\overset{*}{f_n})\right|$$

Fix $n \geq n_0$ and let $m \to \infty$ in the previous inequality. We get

$$\|f_n\|_2^2 \leq \|f_n\|_2 \cdot \varepsilon ,$$

so $$\|f_n\|_2 \leq \varepsilon, \quad \forall n \geq n_0$$

and we obtain the conclusion. $\square$

6.3.3 Remark : Apparently, there is a mistake in the previous proof. As an exercise, try to identify it and formulate an appropiate Lemma to overcome the difficulty.

6.3.4 The definition of $L^2$

Define $L^2 \equiv \{ F \in \mathcal{P}' :$ there exists $(f_n)$ in $\mathcal{E}$, $\|\cdot\|_2$-Cauchy, such that $F_{f_n} \xrightarrow{\mathcal{P}'} F \}$

Then
1. $L^2$ is a linear subspace of $\mathcal{P}'$
2. Obviously $\mathcal{E} \subset L^2$
3. Introduce an inner product in $L^2$ by
$$\langle F, G \rangle = \lim_{n\to\infty} \langle f_n, g_n \rangle$$
where $(f_n) \subset \mathcal{E}$, $(g_n) \subset \mathcal{E}$ are $\|\cdot\|_2$-Cauchy sequences such that $F_{f_n} \xrightarrow{\mathcal{P}'} F$, $F_{g_n} \xrightarrow{\mathcal{P}'} G$
4. $(L^2, \langle\,\rangle)$ is a complete inner product space, so a Banach space endowed with the norm $\|F\|_2 = \langle F, F\rangle^{1/2}$
5. $\mathcal{E} \subset (L^2, \|\cdot\|_2)$ is a dense subspace.

6.3.5 Remark : most of the properties above can be easily proved, and it is clear from Lemmata 6.3.1, 6.3.2, that $\langle\ \rangle$ defined in 3 is independent of the particular sequences $(f_n)$ and $(g_n)$.
For example, assume $(f_n), (f'_n), (g_n), (g'_n)$ are $\|\cdot\|_2$-Cauchy sequences in $\mathcal{E}$, $F_{f_n} \xrightarrow{\mathcal{P}'} F$, $F_{f'_n} \xrightarrow{\mathcal{P}'} F$, $F_{g_n} \xrightarrow{\mathcal{P}'} G$, $F_{g'_n} \xrightarrow{\mathcal{P}'} G$. Then, by Lemma 6.3.2, $\|f_n - f'_n\|_2 \to 0$, $\|g_n - g'_n\|_2 \to 0$.
Now $|\langle f_n, g_n\rangle - \langle f'_n, g'_n\rangle| = |\langle f_n - f'_n, g_n\rangle + \langle f'_n, g_n\rangle - \langle f'_n, g'_n\rangle| \leq$
$\leq \|f_n - f'_n\|_2 \cdot \|g_n\|_2 + \|f'_n\|_2 \cdot \|g_n - g'_n\|_2 \to 0$ as $n \to \infty$. $\square$

6.3.6 Remark $(\mathcal{E}, \|\cdot\|_2)$ is a separable normed space (the reason is the following: $(\mathcal{E}, \|\cdot\|_\infty)$ is already separable, because of the Weierstrass approximation Theorem; now recall that $\|\cdot\|_2 \leq \|\cdot\|_\infty$). It follows that $(L^2, \langle\,\rangle)$ is a separable Hilbert space.

## Lemma

Let $(f_n)_1^\infty$ be a $\|\cdot\|_2$-Cauchy sequence in $\mathcal{E}$. Assume $F_{f_n} \xrightarrow{\mathcal{P}'} 0$. Then $\langle f, f_n\rangle \longrightarrow 0$, $\forall f \in \mathcal{E}$

Proof: We proved before that $\mathcal{P}$ is dense in $(\mathcal{E}, \|\cdot\|_\infty)$. Then, given $\varepsilon > 0$ we can find $u \in \mathcal{P}$ such that $\|f-u\|_\infty < \varepsilon$. Now

$$|\langle f, f_n\rangle| = |\langle u, f_n\rangle + \langle f-u, f_n\rangle| \leq |F_{f_n}(u^*)| + \|f-u\|_2 \cdot \|f_n\|_2 \leq$$
$$\leq |F_{f_n}(u^*)| + \|f-u\|_\infty \cdot M < |F_{f_n}(u^*)| + \varepsilon . M$$

where $\|f_n\|_2 \leq M$, $\forall n \in \mathbb{N}$.

Now it is enough to let $n \to \infty$ and observe that $F_{f_n}(u^*) \longrightarrow 0$ as $n \longrightarrow \infty$. □

## Exercises

1. Is $\delta_{x_0} \in L^2$ for some $x_0 \in \mathbb{R}$ ?
2. Prove that $\|f\|_2 = \sup\{\langle f, g\rangle : g \in \mathcal{P}, \|g\|_2 \leq 1\}$, $\forall f \in \mathcal{E}$
3. Is $\mathcal{P}$ dense in $(L^2, \|\cdot\|_2)$ ?
4. We provided in 6.2.2 a sequence of functions in $\mathcal{E}$ such that $(f_n)$ did not converge in $(\mathcal{E}, \|\cdot\|_2)$ although it was a $\|\cdot\|_2$-Cauchy sequence. It follows that $\|\cdot\|_2$-converges to some element $F \in L^2 \subset \mathcal{P}'$. Identify how this element acts on functions $u \in \mathcal{P}$.
5. Prove that $(L^2, \|\cdot\|_2)$ is complete.
6. Prove that, in $\mathcal{E}$, we have $\|\cdot\|_2 \leq \|\cdot\|_\infty$.
7. Prove that $\|\cdot\|_2$ is a norm in $\mathcal{E}$.
8. Prove that $\|F\|_2 = \sup\{|F(u)| : u \in \mathcal{P}, \|u\|_2 \leq 1\}$, $\forall F \in L^2$.

## Summary on Hilbert spaces

$[\mathbb{H}, (\cdot,\cdot)]$ : Hilbert space, $(\ )$ is an inner product, $\mathbb{H}$ a vector space.

① $(\alpha u, v) = \alpha(u,v) = (u, \alpha^* v)$
② $(u_1+u_2, v) = (u_1, v) + (u_2, v)$
③ $(u, v_1+v_2) = (u, v_1) + (u, v_2)$
④ $(u,v) = (v,u)^*$
⑤ $(u,u) \geq 0$, $(u,u) = 0 \Longleftrightarrow u = 0$

Define $\|u\|_2 = (u,u)^{1/2}$. It is a norm on $\mathbb{H}$.
Then $(\mathbb{H}, \|\cdot\|_2)$ is a normed space. We add the requirement that it should be a Banach space, i.e., it should be complete.

Cauchy-Schwarz inequality

$(u,v) \leq \|u\|_2 . \|v\|_2$

Orthogonality $\quad u \perp v \equiv (u,v) = 0$

Parallelogram identity : $\|u+v\|^2 + \|u-v\|^2 = 2\|u\|^2 + 2\|v\|^2$

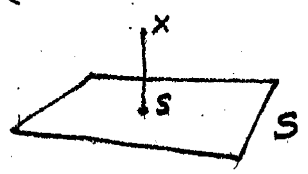
Remark: a consequence of the paral. identity is $\|x-x_0\| \leq R$, $\|y-y_0\| \leq R$, $\left\|\frac{x+y}{2} - x_0\right\| \geq r \Rightarrow \|x-y\| \leq 2\sqrt{R^2-r^2}$

Some properties

Continuity of the inner product : $u_n \to u$, $v_n \to v \Rightarrow (u_n, v_n) \to (u,v)$
It follows that $u \perp S \Rightarrow u \perp \bar{S}$ (remark : $u \perp S$, where $S$ is a subset of $\mathbb{H}$, means $u \perp s, \forall s \in S$)
given $S \subset \mathbb{H}$, $S^\perp \equiv \{s^\perp \in \mathbb{H} : \langle s, s^\perp \rangle = 0, \forall s \in S\}$ is called the orthogonal to $S$. Then $S^\perp$ is a closed subspace.

Orthogonal projection :

given $S \subset \mathbb{H}$ and $x \in \mathbb{H}$, there exists, if $S$ is a closed subspace, a unique element $s \in S$ such that $d(x,S) = \|x-s\|_2$. Moreover, $s$ is the only element in $S$ such that $(x-s) \perp S$. It is called the orthogonal projection of $x$ into $S$




Orthogonal complements :

Given a closed subspace $A \subset \mathbb{H}$, then $A^\perp$ is a closed subspace of $\mathbb{H}$ and $\mathbb{H} = A \oplus A^\perp$, in the sense that $A \cap A^\perp = \{0\}$ and $\forall x \in \mathbb{H}$, $x = a + a^\perp$ such that $a \in A$, $a^\perp \in A^\perp$. $A^\perp$ is called the orthogonal complement to $A$ in $\mathbb{H}$.

Riesz representation Theorem : The dual space of $\mathbb{H}$, a Hilbert space, is isometrically isomorphic to $\mathbb{H}$, in the following sense : Given $x \in \mathbb{H}$, the mapping $\mathbb{H} \to \mathbb{C}$ given by $y \to \langle y, x \rangle$ is linear and continuous, denoted by $L_x$. Reciprocally, given $L \in \mathbb{H}^*$, the dual of $\mathbb{H}$, there exists $x \in \mathbb{H}$ such that $L = L_x$. Moreover, $\|x\|_2 = \|L_x\|$.

## Remark

On $\mathbb{H}$, a Hilbert space, the most important features follow from the Parallelogram identity and its consequences. Precisely

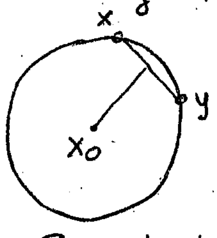
$$\boxed{\text{Given } x_0, x, y \text{ in } \mathbb{H} \text{ such that } x \in B(x_0;R),\ y \in B(x_0;R) \text{ and } \frac{x+y}{2} \notin B(x_0;r), \text{ then } \|x-y\| \leq 2\sqrt{R^2-r^2}}$$

Proof: $\|(x-x_0)+(y-x_0)\|^2 + \|(x-x_0)-(y-x_0)\|^2 = 2\|x-x_0\|^2 + 2\|y-x_0\|^2$

$$\|x+y-2x_0\|^2 + \|x-y\|^2 = 2\|x-x_0\|^2 + 2\|y-x_0\|^2$$

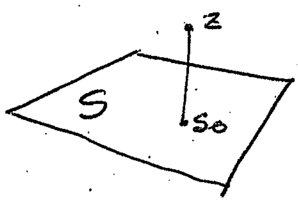
$$\|x-y\|^2 \leq 2R^2 + 2R^2 - 4\left\|\frac{x+y}{2} - x_0\right\|^2 \leq 4(R^2-r^2) \quad \square$$

Geometrically it means that $\frac{x+y}{2}$ far away from $x_0$ forces $x, y$ to be close.




$$\boxed{\text{Given } S, \text{ a closed subspace of } \mathbb{H}, \text{ and } z \in \mathbb{H}; \text{ there exists a unique nearest point to } z \text{ in } S.}$$

Proof: Let $d := \text{dist}(z, S)$ and choose a sequence $(s_n)$ in $S$,




$$d \leq \|z - s_n\| < d + 1/n, \ \forall n$$

$(s_n)$ is Cauchy: Let $n < m$.

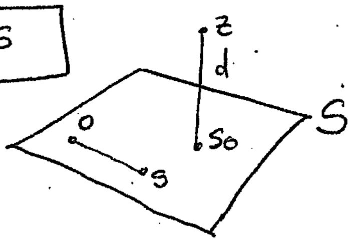
$$\left.\begin{array}{l} \|s_n - z\| \leq d + 1/n \\ \|s_m - z\| \leq d + 1/n \\ \left\|\frac{s_n+s_m}{2} - z\right\| \geq d \end{array}\right\} \quad \|s_n - s_m\| \leq 2\sqrt{(d+1/n)^2 - d^2} \longrightarrow 0$$

It follows that $s_n \to s_0$ and $d = \|z - s_0\|$.

Assume $s_0, s$ in $S$ satisfy $\|s_0 - z\| = \|s - z\| = d$, $s \neq s_0$

It follows that $\left\|\frac{s_0+s}{2} - z\right\| < d$, a contradiction $\square$

$$\boxed{\text{In the situation of the former result, } (z-s_0) \perp S}$$




Proof: $\|(z-s_0) - as\|^2 = \|z - (s_0 + as)\|^2 \geq \|z - s_0\|^2$

$\langle z - s_0 - as, z - s_0 - as\rangle = \|z - s_0\|^2 - \langle z - s_0, as\rangle$

$- \langle as, z - s_0\rangle + |a|^2\|s\|^2 \geq \|z - s_0\|^2$

Then $|a^2| \cdot \|s\|^2 - \langle z - s_0, as\rangle - \langle as, z - s_0\rangle \geq 0$

Let $a = \langle z - s_0, s\rangle \cdot \|s\|^{-2}$

We easily get $\quad -|\langle s, z - s_0\rangle|^2 \cdot \|s\|^{-2} \geq 0$, so $\langle s; z - s_0\rangle = 0 \quad \square$

## Orthonormal bases

$\mathbb{H}$ : Hilbert space.

$\{e_n : n \in \mathbb{N}\}$ a sequence in $\mathbb{H}$ is called an orthonormal sequence if $\|e_n\| = 1, \forall n$ and $\langle e_n, e_m \rangle = \delta_{nm}, \forall n, m$.

It is called an orthonormal basis if, moreover, $\mathrm{lin}(e_n)$ is dense in $\mathbb{H}$ (where $\mathrm{lin}(e_n)$ means the linear hull of $(e_n)$). (dense means $\overline{\mathrm{lin}}(e_n) = \mathbb{H}$). It is equivalent to say that $(e_n)$ is maximal, in the sense that not larger sequence can be orthonormal.

## Gram-Schmidt orthogonalization method :

Given $\{U_n\}$ a sequence in $\mathbb{H}$, there is an orthonormal sequence $(e_n)$ in $\mathbb{H}$ such that $u_n \in \mathrm{lin}\{e_1, \dots, e_n\}, \forall n$.

## Existence of orthonormal bases

Let $\mathbb{H}$ be a separable Hilbert space. Then there exits a sequence $(e_n)$ which forms an orthonormal basis.

## Orthogonal expansions (Fourier Series)

Let $\mathbb{H}$ be a Hilbert space (separable) and $(e_n)_1^\infty$ an orthonormal (basis) sequence. Given $x \in \mathbb{H}$, we have

$$x = \sum_{n=1}^{\infty} \langle x, e_n \rangle . e_n$$

in the sense of $\|\cdot\|_2$, i.e., $\|x - \sum_{n=1}^{k} (x, e_n) e_n \|_2 \to 0$ when $k \to \infty$.

The coefficients $(x, e_n)$ are called Fourier coefficients of $x$ in the basis $(e_n)$, and the series $\sum (x, e_n) e_n$ the Fourier series of $x$ in the basis $(e_n)$.

Remark : Assume that the basis has indices in $\mathbb{Z}$, i.e., we consider $(e_n)_{-\infty}^{+\infty}$. Then we have

$$x = \sum_{-\infty}^{+\infty} (x, e_n) e_n$$

in the sense $\|x - \sum_{-k}^{k} (x, e_n) e_n \|_2 \to 0$ when $k \to +\infty$.

In both cases we shall write $x \sim \sum (x, e_n) e_n$ to indicate that $x$ has $\sum (x, e_n) e_n$ as its Fourier Series.

**Theorem**: Let $\mathbb{H}$ be a Hilbert space (separable), $(e_n)$ an orthonormal basis in $\mathbb{H}$. Then, given $x \in \mathbb{H}$, denote $a_n := (x, e_n)$ its $n$-th Fourier coefficient. Then

① $\Sigma |a_n|^2 < \infty$

② $\|x\|_2^2 = \Sigma |a_n|^2$

③ Given $(b_n)$ in $\mathbb{C}$ such that $\Sigma |b_n|^2 < +\infty$, there exists an unique $x \in \mathbb{H}$ such that $(x, e_n) = b_n, \forall n$

Summarizing, the mapping $\mathbb{H} \longrightarrow l_2$ given by $x \rightarrow (x, e_n)_1^\infty$ is a linear isometry between $\mathbb{H}$ and the space $l_2$ of all square-summable sequences with the $\|\cdot\|_2$-norm:

$$\|(a_n)\|_2^2 = \sum_1^\infty |a_n|^2$$

Some examples

- Finite-dimensional examples: Let $\mathbb{H} \equiv \mathbb{C}^n = \{(c_1, c_2, \ldots, c_n) : c_i \in \mathbb{C}, i = 1, 2, \ldots n\}$. The inner product is
$$(\mathbf{c}, \mathbf{d}) = \sum_{i=1}^n (c_i \cdot d_i^*) = \sum_{i=1}^n c_i d_i^*$$
- Infinite-dimensional examples (separable). By the last theorem, there is only one example:
  - $l_2 \equiv \{\mathbf{c} = (c_n)_1^\infty, c_i \in \mathbb{C}, i = 1, 2, \ldots\}$
    the inner product is $(\mathbf{c}, \mathbf{d}) = \sum_1^\infty c_i d_i^*$
  - Some isometric examples:
    - $L^2 \equiv \{f : \mathbb{R} \rightarrow \mathbb{C} :$ Lebesgue-measurable functions such that $f^2$ is Lebesgue-integrable$\}$.
    The inner product is $(f, g) = \int_{\mathbb{R}} f(x) \cdot g^*(x) dx$.
  - $l_2(\mathbb{Z}) \equiv \{\mathbf{c} = (c_n)_{-\infty}^{+\infty}, c_i \in \mathbb{C}, i \in \mathbb{Z}\}$
    the inner product is $(\mathbf{c}, \mathbf{d}) = \sum_{-\infty}^{+\infty} c_i d_i^*$.
  - $L^2$ in $\mathcal{P}'$; it was defined as the space of all periodic distributions $F$ such that there exists a sequence $(u_n)_1^\infty$ in $\mathcal{E}$, the continuous periodic functions, such that $(u_n)$ is $\|\cdot\|_2$-Cauchy and $u_n \xrightarrow{\mathcal{P}'} F$ (we identify elements in $\mathcal{E}$ with periodical distributions $u \leftrightarrow F_u$).

Every time we shall write $\boxed{L^2}$ we consider this last example.
The inner product is given by

$$(f, g) = \frac{1}{2\pi} \int_0^{2\pi} f(x) g^*(x) dx, \text{ if } f, g \in \mathcal{E}$$

$$(F, G) = \lim_n (f_n, g_n), \text{ if } F, G \in L^2 \text{ and } f_n \xrightarrow{\|\cdot\|_2} F, g_n \xrightarrow{\|\cdot\|_2} G$$
where $f_n \in \mathcal{E}, g_n \in \mathcal{E}$.

## An orthonormal basis in $L^2$

Observe that $\mathcal{P} \subset \mathcal{C} \subset L^2$. The family $(e_n)_{n=-\infty}^{n=+\infty}$ given by
$$e_n(x) = \exp(inx), \quad n \in \mathbb{Z}$$
is an orthonormal basis in $L^2$

<u>Proof</u>: $\|e_n\|_2^2 = \frac{1}{2\pi}\int_0^{2\pi} e_n(x)\, e_n^*(x)\,dx = \frac{1}{2\pi}\int_0^{2\pi} dx = 1, \ \forall n$

$$(e_n, e_m) = \frac{1}{2\pi}\int_0^{2\pi} e_n(x)\, e_m^*(x)\,dx = \frac{1}{2\pi}\int_0^{2\pi} e^{inx} \cdot e^{-imx}\,dx =$$
$$= \frac{1}{2\pi}\int_0^{2\pi} e^{ix(n-m)}\,dx = \left(\frac{1}{i(n-m)}\right) \cdot e^{ix(n-m)}\Big|_{x=0}^{x=2\pi} = 0$$
if $n \neq m$.

Hence it is an orthonormal sequence in $L^2$. We shall prove that it is moreover a basis. This is a consequence of the following two facts: We proved (Weierstrass approximation theorem) that

$\forall u \in \mathcal{C}$, there exists a sequence $(v_n)$ in $\mathcal{P}$ such that $v_n \xrightarrow{\|\cdot\|_\infty} u$. In fact, each $v_n$ is a trigonometric polynomial. and we know that $\|\cdot\|_2 \leq \|\cdot\|_\infty$ on $\mathcal{C}$, so $v_n \xrightarrow{\|\cdot\|_2} u$.
Moreover, given $F \in L^2$, there exists a sequence $(u_n)$ in $\mathcal{C}$ such that $u_n \xrightarrow{\|\cdot\|_2} F$. $\square$

Now we can apply on $L^2$ the complete theory of Hilbert spaces, as $L^2$ is a (separable) Hilbert space with an inner product as before and $(e_n)_{-\infty}^{+\infty}$ an orthonormal basis.

<u>Remark</u>: $L^2 \subset \mathcal{P}'$. Each $e_n \in \mathcal{P}$. Given $F \in L^2$, its $n$-th Fourier coefficient is given by
$$(F, e_n) \quad \text{(as always, ( ) is the inner product)}$$
and it coincides with $\langle e_n^*, F\rangle$ (as always, $\langle\ \rangle$ is the bilinear form of the duality $\langle \mathcal{P}, \mathcal{P}'\rangle$)

Then we have $\quad (F, e_n) = \langle e_n^*, F\rangle = \langle e_{-n}, F\rangle$

$\forall n \in \mathbb{Z}, \forall F \in \mathcal{P}', F \in L^2$

Given $F, G$ in $\mathcal{P}'$, $F \sim \sum_{-\infty}^{\infty} a_n e_n$, $\quad G \sim \sum_{-\infty}^{+\infty} b_n e_n$

then $\quad (F, G) = \sum a_n b_n^*$

## A remark on convergence

Let $F \in L^2$, $F \sim \sum_{-\infty}^{\infty} a_n e_n$ its Fourier series.
By the general theory of Hilbert spaces we have

$$\left\| F - \sum_{-N}^{N} a_n e_n \right\|_2 \longrightarrow 0 \quad \text{when } N \to +\infty$$

In particular, this is true for $F = F_u$, where $u \in \mathcal{C}$.
Choose $x \in [0, 2\pi]$. We may ask whether $\sum_{-N}^{N} a_n e_n(x) = \sum_{-N}^{N} a_n \exp(inx)$ converges to $u(x)$.

This is not always true. The following Theorem gives a particular instance where this is true, even in a very strong sense: we shall have uniform convergence:

> **Theorem**: Consider $F \in L^2$ such that $F \sim \sum_{-\infty}^{\infty} a_n e_n$ and $(a_n) \in \ell_1$ (i.e. $\sum |a_n| < \infty$). Then the partial sums $\left( \sum_{-N}^{N} a_n e_n \right)_N$ converge uniformly to a function (therefore continuous function) $u$, so that $F = F_u$, and the Fourier coefficients of $F$ are $(a_n)$.

**Proof**: Let $u_N \equiv \sum_{-N}^{N} a_n e_n$, $\forall N$. Now, given $N \leq M$,

$$\| u_N - u_M \|_\infty = \left\| \sum_{N}^{M} a_n e_n + \sum_{-M}^{-N} a_n e_n \right\|_\infty \leq$$

$$\leq \sum_{N}^{M} |a_n| + \sum_{-M}^{-N} |a_n| \longrightarrow 0 \quad \text{when } N, M \to +\infty.$$ It follows that

$(u_N)$ is a $\|\cdot\|_\infty$-Cauchy sequence in $\mathcal{C}$, and we know that the uniform limit of a sequence of continuous functions is itself continuous. Then $(u_N) \xrightarrow{\|\cdot\|_\infty} u \in \mathcal{C}$. In particular $(u_N) \xrightarrow{\|\cdot\|_2} u$ (and we knew from the very beginning that $(u_N) \xrightarrow{\|\cdot\|_2} F$, so $u = F$ and the Fourier coefficients of $F = F_u$ are $(a_n)_{n=-\infty}^{+\infty}$). $\square$

> **Corollary**: Let $u \in \mathcal{C}$ and assume $u$ is continuously differentiable (i.e., $u$ has a continuous derivative). Then $u \sim \sum a_n e_n$ where $(a_n) \in \ell_1$, so the partial sums, by the previous theorem, converge uniformly to $u$. Moreover, if $u \in \mathcal{P}$, they converge in the sense of $\mathcal{P}$.

**Proof:** consider $u \in \mathcal{E} \sim \sum_{-\infty}^{\infty} a_n e_n$, where $a_n = (u, e_n) =$
$= \frac{1}{2\pi}\int_0^{2\pi} u(x) e_n^*(x)\,dx = \frac{1}{2\pi}\int_0^{2\pi} u(x)\, \underline{e}_n(x)\,dx$

Consider now $Du \in \mathcal{E} \sim \sum b_n e_n$. then $b_n = (Du, e_n) =$
$$= \frac{1}{2\pi}\int_0^{2\pi} Du(x)\underline{e}_n(x)\,dx = \frac{in}{2\pi}\int_0^{2\pi} u(x)\underline{e}_n(x)\,dx = in a_n$$
It follows that we have the following relationship between the u's and Du's coefficients:
$$\boxed{b_n = in a_n, \ \forall n}$$
Now
$$\sum |a_n| = |a_0| + \sum_{n\neq 0} |a_n| = |a_0| + \sum_{n \neq 0} \frac{1}{n}|b_n| \leq$$
$$\leq |a_0| + \left(\sum_{n\neq 0}\frac{1}{n^2}\right)^{1/2}\left(\sum_{n\neq 0}|b_n|^2\right)^{1/2} = |a_0| + K\|Du\|_2$$
and we can appy the former theorem.

Suppose now that $u \in \mathcal{P}$. Let $u_N := \sum_{-N}^{N} a_n e_n$, where $a_n = (u, e_n), \forall n$.
It follows that $Du_N = \sum_{-N}^{N} b_n e_n$, and, in general, $D^k u_N$ is the N-th partial sum of the Fourier series of $D^k u$. Apply the first part of the Corollary (already proved) and we get that $D^k u_N \to D^k u$ in $\|\cdot\|_\infty$. This is the same that $u_N \xrightarrow{\mathcal{P}} u$. $\square$

## Identification of the sequence of Fourier coefficients

Given $F \in L^2$, we associated a sequence $(a_n)_{\mathbb{Z}}$ of Fourier coefficients and formed the Fourier series $\sum a_n e_n$. We wrote

$$F \sim \sum_{-\infty}^{+\infty} a_n e_n$$

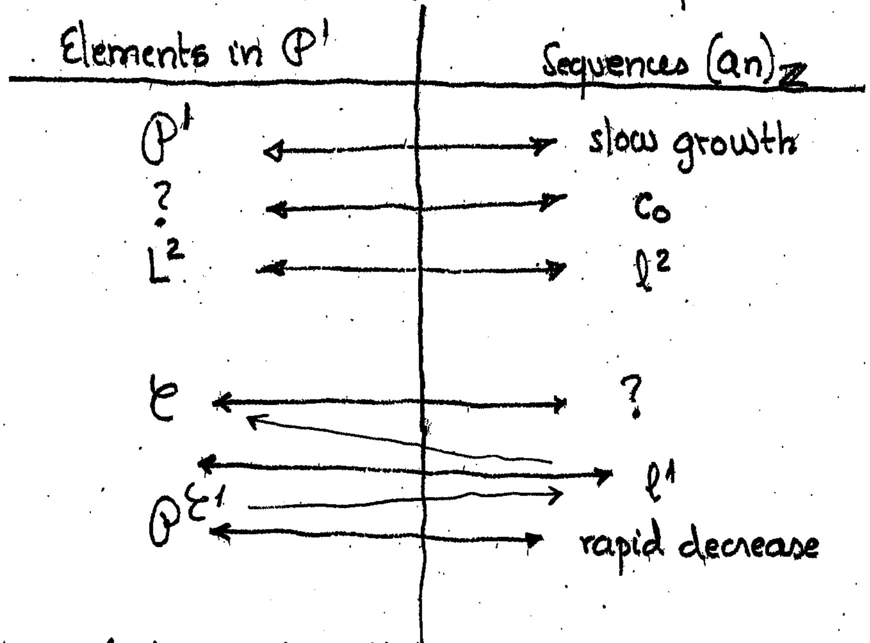
and we saw that $(a_n) \in \ell_2(\mathbb{Z})$ (and conversely, every sequence $(a_n) \in \ell_2(\mathbb{Z})$ is the sequence of Fourier coefficients of a given $F \in L^2$. We had, moreover, $\sum_{-N}^{N} a_n e_n \xrightarrow{\|\cdot\|_2} F$, $\sum |a_n|^2 = \|F\|_2^2$, and

$$a_n = \langle \underline{e}_n, F \rangle, \ \forall n$$

Now consider the duality $\langle \mathcal{P}, \mathcal{P}' \rangle$ and just recall that $e_n \in \mathcal{P}$, $\forall n \in \mathbb{Z}$. Obviously we can extend the definition of Fourier coefficients for every $F \in \mathcal{P}'$: define

$$\boxed{a_n := \langle \underline{e}_{n}, F \rangle, \ \forall n \in \mathbb{Z}}$$

and consider the (formal) series $\sum_{-\infty}^{\infty} a_n e_n$. We get a sequence $(a_n)_{-\infty}^{+\infty}$. The problem is to try to be able to recognize which elements in $\mathcal{P}'$ correspond to which sequences $(a_n)_{\mathbb{Z}}$.




**Definitions**: $(a_n)_{\mathbb{Z}}$ is of <u>rapid decrease</u> $\equiv$ $\forall r > 0 \ \exists c(r) > 0$ such that $|a_n| \leq c.|n|^{-r}$, $\forall n \in \mathbb{Z}, n \neq 0$

$(a_n)_{\mathbb{Z}}$ is of <u>slow growth</u> $\equiv$ $\exists r > 0, \exists c > 0$ such that $|a_n| \leq c.|n|^{r}$, $\forall n \in \mathbb{Z}, n \neq 0$

Theorem

$$\boxed{u \in \mathcal{P} \Longleftrightarrow (a_n)_{\mathbb{Z}} \text{ is of rapid decrease}}$$

Proof: ($\Rightarrow$). Assume $u \in \mathcal{P}$. Consider $D^k u$. We know that $D^k u \sim \sum (in)^k a_n e_n$ and then $\sum_n |in|^{2k} |a_n|^2 = \|D^k u\|_2^2$, so, in particular

$$|in|^k . |a_n| \leq \|D^k u\|_2$$

and then

$$|n|^k . |a_n| \leq \|D^k u\|_2, \ \forall n$$

This proves that $(a_n)$ is of rapid decrease.

($\Leftarrow$) Let $(a_n)_{\mathbb{Z}}$ be of rapid decrease. Define

$$u_N(x) \equiv \sum_{-N}^{N} a_n \exp(inx) \ ; \ N \in \mathbb{N} \cup \{0\}.$$

Take $r=2$. We get $\sum |a_n| < \infty$. It follows that $u_N \xrightarrow{\|\cdot\|_\infty} u \in \mathcal{C}$. and $u$ has, as the sequence of Fourier coefficients, precisely $(a_n)$.

Observe that $$D^k u_N(x) = \sum_{-N}^{N} a_n (in)^k \exp(inx), \ N \in \mathbb{N} \cup \{0\}.$$

The same argument proves that $D^k u_N \xrightarrow{\|\cdot\|_\infty} f_k \in \mathcal{C}$, and $f_k$ has $(a_n (in)^k)_{n \in \mathbb{Z}}$ as the sequence of Fourier coefficients.

Fix $x \in [0, 2\pi]$. Using that the convergence above is uniform we get

$$\int_0^x D^k u_N(t)\,dt \xrightarrow[N]{} \int_0^x f_k(t)\,dt$$

so

$$D^{k-1} u_N(x) - D^{k-1} u_N(0) \xrightarrow{N} \int_0^x f_k(t)\,dt$$

We know that $$D^{k-1} u_N(x) \xrightarrow{N} f_{k-1}(x).$$

We get

$$f_{k-1}(x) - f_{k-1}(0) = \int_0^x f_k(t)\,dt$$

and then $$f'_{k-1}(x) = f_k(x), \ \forall k = 1, 2, 3, \ldots, \ \forall x.$$

By finite induction, we get that $u \in \mathcal{P}$ and $u_N \xrightarrow{\mathcal{P}} u$. $\square$

**Theorem** $\boxed{F \in \mathcal{P}' \Longleftrightarrow (a_n) \text{ is of slow growth}}$

**Proof**: $(\Rightarrow)$ Let $F \in \mathcal{P}'$. We know that $F$ is of some order, say $k$. Then $|F(u)| \leq c(\|u\|_\infty + \|Du\|_\infty + \ldots + \|D^k u\|_\infty)$, $\forall u \in \mathcal{P}$, for some $c > 0$.

In particular
$$|F(e_{-n})| = |a_n| \leq c(1 + |n| + \ldots + |n|^k) < 2c|n^k|$$
and so $(a_n)$ is of slow growth.

$(\Leftarrow)$ Suppose $(a_n)$ is of slow growth. Then $\exists r > 0, \exists c > 0$ such that
$$|a_n| \leq c|n|^r, \quad \forall n \in \mathbb{Z}, n \neq 0$$
Choose $k > 0$ such that $r \leq k-2$. We get
$$|a_n| \leq c|n|^{k-2}, \quad n \neq 0$$
Let $b_0 = 0$, $\quad b_n = (in)^{-k} a_n$, $n \neq 0$. Let
$$V_N := \sum_{-N}^{N} b_n e_n$$
We get $\quad |b_n| \leq c \cdot n^{-2}$, so $\sum |b_n| < +\infty$ and then $V_N \xrightarrow{\|\cdot\|_\infty} V \in \mathcal{C}$

Form the distribution
$$F := D^k F_V + F_f, \quad \text{where } f \equiv a_0 e_0$$
We shall prove that
$$F \sim \sum a_n e_n$$
$(n \neq 0)$ $\quad F(e_{-n}) = D^k F_V(e_{-n}) = F_V[(-D)^k e_{-n}] = (in)^k F_V(e_{-n}) =$
$= (in)^k \cdot b_n = a_n$

$(n = 0)$ $\quad F(e_0) = D^k F_V(e_0) + F_f(e_0) = F_f(e_0) = a_0 \quad \square$

**Remark**: Let $F \in \mathcal{P}'$. By the previous theorem, we get that $(a_n)$ is of slow growth. Then, in the proof we fond $V \in \mathcal{C}$ and a constant function $f$ such that for some $k$,
$$D^k F_V + F_f$$
has $(a_n)$ as the sequence of Fourier coefficients. This means
$$\boxed{F = D^k F_V + F_f}$$
and we recover the fact that <u>every distribution $F$ can be written as the sum of a derivative of a function (continuous and periodic) and the distribution of a constant function.</u>

## The use of Fourier Series in calculus

We have the following situation: every $F \in \mathcal{P}'$ has a Fourier Series associated to it. Precisely

$$F \sim \sum_{-\infty}^{+\infty} a_n e_n$$

where $a_n = \langle e_{-n}, F \rangle$, $\forall n \in \mathbb{Z}$.

We say nothing regarding the convergence of this series. We can, at least, say that the sequence $(a_n)$ identifies $F$, in the sense that different $F$'s have different sequences $(a_n)$.
(the reason being that $\{e_n : n \in \mathbb{Z}\}$ is linearly dense in $\mathcal{P}$ when this last space has the topology associated to what we called convergence in $\mathcal{P}'$).

This section deals with the following question: we defined several operations with distributions. Is it possible to develop a calculus using Fourier coefficients which reproduces all the operations introduced?

### (A) The action of $F \in \mathcal{P}'$ on $u \in \mathcal{P}$

Let $F \sim \sum a_n e_n$, $u \sim \sum b_n e_n$ (recall that $u \in \mathcal{P} \subset \mathcal{P}'$).
Then $\langle u, F \rangle = \sum a_n b_{-n}$

**Proof**: We know that $\sum_{-N}^{N} b_n e_n \xrightarrow{\mathcal{P}} u$. Then $\langle \sum_{-N}^{N} b_n e_n, F \rangle \to \langle u, F \rangle$. Now, $\langle \sum_{-N}^{N} b_n e_n, F \rangle = \sum_{-N}^{N} b_n \langle e_n, F \rangle =$
$= \sum_{-N}^{N} a_n b_n = \sum_{-N}^{N} a_n b_{-n}$. It follows that $\langle u, F \rangle = \sum a_n b_{-n}$. □

**Corollary**: Let $F \in \mathcal{P}'$, $F \sim \sum a_n e_n$. Then $\sum_{-N}^{N} a_n e_n \xrightarrow{\mathcal{P}'} F$.

**Proof**: Given $u \in \mathcal{P}$, $\langle u, \sum_{-N}^{N} a_n e_n \rangle = \sum_{-N}^{N} a_n \langle u, e_n \rangle =$
$= \sum_{-N}^{N} a_n b_{-n} \to \sum_{-\infty}^{\infty} a_n b_{-n} = \langle u, F \rangle$, where $u \sim \sum b_n e_n$. □

### (B) The convolution

Recall that given $F, G$ in $\mathcal{P}'$, $u \in \mathcal{P}$, we defined convolution in two steps. First, we defined

$$(G * u)(x) = G[T_x \check{u}]$$

Then

$$(F * G)(u) = F[\tilde{G} * u]$$

We have the following

**Theorem** $F \sim \sum a_n e_n$ , $F \in \mathcal{P}'$
$G \sim \sum b_n e_n$ , $G \in \mathcal{P}'$
$u \sim \sum c_n e_n$ , $u \in \mathcal{P}$
Then $G * u \sim \sum (b_n c_n) e_n$
$F * G \sim \sum (a_n b_n) e_n$

**Proof:** $u(x) = \sum c_n \exp(inx)$

$$(T_x \tilde{u})(y) = \tilde{u}(y-x) = u(x-y) = \sum c_n \exp(in(x-y)) = \sum c_n \exp(inx) \cdot \exp(-iny) =$$
$$= \sum c_{-n} \exp(-inx) \exp(iny).$$

In order to calculate $G[T_x \tilde{u}]$ just apply the previous result in (A)

$$G[T_x \tilde{u}] = \sum b_n c_n \exp(inx).$$

Observe now that $\tilde{G}(u) = G(\tilde{u}) = \sum b_n c_n \Rightarrow \tilde{G} \sim \sum b_{-n} e_n$

Then $\tilde{G} * u \sim \sum b_{-n} c_n e_n$

$$F * G\,(u) = F[\tilde{G} * u] = \sum a_n b_n c_{-n}$$

hence $F * G \sim \sum (a_n b_n) e_n$. □

(C) <u>Derivation</u>

Let $u \in \mathcal{P}$, $u \sim \sum_{-\infty}^{+\infty} a_n e_n$. We know that $\sum_{-N}^{N} a_n e_n \xrightarrow{\mathcal{P}} u$. In particular, this means that $D^k(\sum_{-N}^{N} a_n e_n) \xrightarrow{\|\cdot\|_\infty} D^k u$. Now

$$D^k\left(\sum_{-N}^{N} a_n \exp(inx)\right) = \sum_{-N}^{N} a_n (in)^k \exp(inx) = \sum_{-N}^{N} a_n (in)^k e_n$$

It follows that $D^k u \sim \sum_{-\infty}^{\infty} a_n (in)^k e_n$

Let $F \in \mathcal{P}'$, $F \sim \sum_{-\infty}^{\infty} b_n e_n$

$$D^k F(u) = F((-1)^k D^k u) = (-1)^k F[D^k u] = (-1)^k \sum b_n a_{-n} (-1)^k (in)^k$$
$$= \sum b_n a_n (in)^k$$

Then $D^k F \sim \sum_{-\infty}^{\infty} (in)^k b_n e_n$

(D) <u>Limits</u>

**Theorem:** Assume $(u_m)_1^\infty$ a sequence in $\mathcal{P}$, $u_m \sim \sum_{n=-\infty}^{\infty} a_{mn} e_n$, $\forall m$

Then $u_m \xrightarrow[m]{\mathcal{P}} u \in \mathcal{P} \iff a_{mn} \xrightarrow[m]{} a_n$, $\forall n$
and $\forall r > 0 \ \exists\, c(r) > 0$ such that
$|a_{m,n}| \leq c(r) |n|^{-r}$, $\forall m$, $\forall n \neq 0$

$(\Leftarrow)$

Proof : Obviously, $|a_n| \leq c(r).|n|^{-r}$, $\forall n \neq 0$. It follows that $(a_n)$ is the sequence of Fourier coefficients of a function $u \in \mathcal{P}$. We shall prove that $u_m \xrightarrow{\mathcal{P}} u$.

$$|(u_m - u)(x)| = \left| \sum_{n=-\infty}^{\infty} (a_{mn} - a_n) \exp(inx) \right|$$

Given $\varepsilon > 0$, choose $N \in \mathbb{N}$ such that $\left( \sum_{N+1}^{\infty} n^{-2} \right) . c(r) < \varepsilon$

Then

$$\left| \sum_{-\infty}^{\infty} (a_{mn} - a_n) \exp(inx) \right| \leq \sum_{-N}^{N} |a_{mn} - a_n| + \sum_{N+1}^{\infty} |a_{mn}| + \sum_{-\infty}^{-N-1} |a_{mn}| +$$
$$+ \sum_{N+1}^{\infty} |a_n| + \sum_{-\infty}^{-N-1} |a_n| < 4\varepsilon + \sum_{n=-N}^{N} |a_{mn} - a_n| = [1]$$

It is enough to choose $m$ big enough to get $[1] < 5\varepsilon$

It follows that $u_m \xrightarrow{\|\cdot\|_\infty} u$.

The argument to prove that $D^k u_m \xrightarrow{\|\cdot\|_\infty} D^k u$ is similar. $\square$

$(\Rightarrow)$ Assume $u_m \xrightarrow[m]{\mathcal{P}} u$

By definition $\quad D^k u_m \xrightarrow[m]{\|\cdot\|_\infty} D^k u, \quad k = 0, 1, 2, \ldots$

Fix $k \in \{0, 1, 2, \ldots\}$. The sequence $\{D^k u_m : m = 1, 2, \ldots\}$ is $\|\cdot\|_\infty$-bounded.

Then we can find $c(k) > 0$ such that

$$\| D^k u_m \|_\infty \leq c(k), \ \forall m$$

As $\| e_n \|_\infty = 1$, $\forall n$, we get

$$|\langle D^k u_m, e_n \rangle| \leq c(k), \ \forall m, \forall n \neq 0$$

$$D^k u_m \sim \sum_{n=-\infty}^{+\infty} a_{mn} e_n (in)^k$$

and we get $\quad |a_{mn} (in)^k| \leq c(k), \ \forall m, n \neq 0$

$$|a_{mn}| \leq c(k) |n|^{-k}, \ \forall m, \forall n \neq 0. \quad \square$$

**Theorem** Assume $(F_m)$ in $\mathcal{P}'$, $F_m \sim \sum_{n=-\infty}^{+\infty} a_{mn} e_n$, $\forall m$

Assume $\exists r > 0$, $\exists c > 0$ such that

$$|a_{mn}| \leq c |n|^r, \ \forall m, \forall n \neq 0, \text{ and } a_{mn} \xrightarrow[m]{} a_n, \ \forall n$$

Then $(a_n)$ is the sequence of Fourier coefficients of a distribution $F \in \mathcal{P}'$ and $F_m \xrightarrow{\mathcal{P}'} F$

$(F_m)_{m=1}^{\infty}$ a sequence in $\mathcal{P}'$, $F_m \sim \sum_{\mathbb{Z}} a_{mn} e_n$, $a_{mn} \xrightarrow[m]{} a_n$, $\forall n$

Assume that $\exists r>0, \exists c>0, |a_{mn}| \leq c|n|^r, \forall m, \forall n \neq 0$

Then there exists $F \in \mathcal{P}'$, $F \sim \sum_{\mathbb{Z}} a_n e_n$ and $F_m \xrightarrow{\mathcal{P}'} F$

Proof: We get $|a_n| \leq c|n|^r$, $\forall n \neq 0$, so $(a_n)$ is the sequence of Fourier coefficients of a periodic distribution $F \in \mathcal{P}'$.

Take $k \in \mathbb{N}$ such that $k \geq r+2$. Let, $\forall m$

$$b_{m,0} = b_0 = 0.$$

$$b_{m,n} = (in)^{-k} \cdot a_{m,n} \qquad n \neq 0$$

$$b_n = (in)^{-k} a_n \qquad n \neq 0$$

It follows that $(b_{mn})_{n \in \mathbb{Z}}$ is the sequence of Fourier coefficients of some continuous periodic function $V_m \in \mathcal{C}$, and

$$F_m = D^k F_{V_m} + F_{f_m} \quad , f_m = a_{m,0} \cdot e_0$$

Similarly $(b_n)$ is the sequence of Fourier coefficients of some $V \in \mathcal{C}$ and

$$F = D^k F_V + F_f, \qquad f = a_0 e_0$$

Observe that $|b_{mn}| \leq c|n|^{-2}$, $n \neq 0$

$$b_{mn} \xrightarrow[m]{} b_n$$

and we get $V_m \xrightarrow{\|\cdot\|_\infty} V$, $a_{m0} \to a_0$

It follows that $F_m \xrightarrow{\mathcal{P}'} F$. $\square$

## Exercises

① We saw that $\mathcal{P} \subset \mathcal{E} \subset L^2 \subset \mathcal{P}'$.
Let $f : [0,2\pi] \to \mathbb{R}$ be an integrable function (think about Riemann integrable or, more generally, Lebesgue integrable). Proof that $f$ defines a periodic distribution $F_f$ as usual and $F_f \in L^2$.

As an example, find the Fourier coefficients of the following integrable functions

a) $f(x) = 0,\ x \in [0,\pi],\quad f(x) = 1,\ x \in ]\pi, 2\pi]$
b) $f(x) = 0,\ x \in [0,\pi],\quad f(x) = x-\pi,\ x \in ]\pi, 2\pi]$.
c) $f(x) = |x-\pi|,\ x \in [0,2\pi]$
d) $f(x) = (x-\pi)^2,\ x \in [0,2\pi]$
e) $f(x) = x,\ x \in [0,2\pi]$
f) $f(x) = |\cos x|,\ x \in [0,2\pi]$

② Let $u \in \mathcal{E}$, $u_N = \sum_{-N}^{N} \langle u, e_{-n} \rangle e_n$ its N'th partial sum of the Fourier series.

Prove that $$u_N(x) = \frac{1}{2\pi} \int_0^{2\pi} D_N(x-y)\, u(y)\, dy = (D_N * u)(x)$$

where $$D_N(x) = \frac{\sin\left(N+\frac{1}{2}\right)x}{\operatorname{sen} \frac{x}{2}}$$ is called the <u>Dirichlet Kernel</u>

③ Compute de Fourier coefficients of $\delta_{x_0}$ and $D^k \delta_{x_0}$

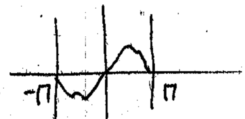
④ Let $(\varphi_m)_1^\infty$ in $\mathcal{E}$ be an approximate identity.

Let $\varphi_m \sim \sum_{n \in \mathbb{Z}} a_{mn} e_n$

Prove that $$\left.\begin{aligned} &|a_{mn}| \leq 1,\ \forall m \in \mathbb{N},\ \forall n \in \mathbb{Z} \\ &\lim_{m\to\infty} a_{mn} = 1,\ \forall n \in \mathbb{Z} \end{aligned}\right\} \quad (*)$$

Do conditions $(*)$ define always an approximate identity?

We take $g \in C[0,\pi]$, $g(0)=g(\pi)=0$.
Extend $g$ to an odd function on $[-\pi,\pi]$, say and then to a $2\pi$-periodic function on $\mathbb{R}$. Define $G = F_g$ and find $(F_t)_{t>0}$ by the previous theorem.

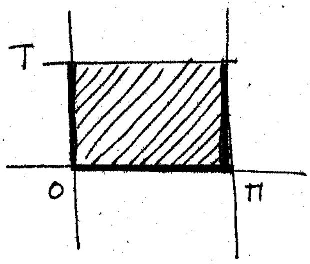



We know $F_t = F_{u_t}$, $u_t \in \mathcal{P}$, $t>0$, $u(x,t) = u_t(x)$, $t>0$, $x \in \mathbb{R}$
It is obvious that $u_t$ is also odd, so $u_t(0) = u_t(\pi) = 0$, $\forall t>0$.
Moreover $t \to u(x,t)$ is infinitely differentiable in $]0,+\infty[$, $\forall x \in \mathbb{R}$.
We shall prove that $u_t \to g$ uniformly on $x \in \mathbb{R}$ as $t \to 0$
It will follow that

$$u(x,t) = \begin{cases} u_t(x), & t>0,\ x \in \mathbb{R} \\ g(x), & t=0 \end{cases}$$

is a classical solution.
In order to prove $u_t \xrightarrow[t\to 0]{\|\cdot\|_\infty} g$, consider the following

**Theorem**: Let $u : [0,\pi] \times [0,+\infty[$ be a classical solution of (HC) (1) and (2). Then, given $T>0$, the maximum value of $u$ in $[0,\pi]\times[0,T]$ is attained on one of the three edges $t=0$, $x=0$, $x=\pi$




Proof: Let $\varepsilon>0$ and define

$$v(x,t) = u(x,t) - \varepsilon t, \quad (x,t) \in [0,\pi]\times[0,T].$$

Obviously $v$ attains the maximum on one of the three edges $t=0$, $x=0$, $x=\pi$. Otherwise the maximum should be attained at $(x_0,t_0)$, $0<x_0<\pi$, $t_0>0$. Then

$$\frac{\partial}{\partial t} v(x_0 t_0) \geq 0$$

$$\frac{\partial}{\partial t} v(x,t) = \frac{\partial}{\partial t} u(x,t) - \varepsilon = \frac{\partial^2}{\partial x^2} u(x,t) - \varepsilon = \frac{\partial^2}{\partial x^2} v(x,t) - \varepsilon$$

Then
$$\frac{\partial^2}{\partial x^2} v(x_0,t_0) = \frac{\partial}{\partial t} v(x_0,t_0) + \varepsilon \geq \varepsilon > 0$$

and $(x_0 t_0)$ is not a maximum for $x \to v(x,t_0)$ on $]0,\pi[$.
Let $M$ be the maximum of $u$ on $t=0$, $x=0$, $x=\pi$ in $[0,\pi]\times[0,T]$.

$$u(x,t) \leq v(x,t) + \varepsilon T \leq \sup_{[0,\pi]\times[0,T]} v + \varepsilon T \leq \sup_{[0,\pi]\times[0,T]} u + 2\varepsilon T = M + 2\varepsilon T$$

Then $\sup_{[0,\pi]\times[0,T]} u \leq M + 2\varepsilon T$. As $\varepsilon>0$ is arbitrary,

$$\sup_{[0,\pi]\times[0,T]} u = M. \quad \square$$

**Theorem**: given $g \in C[0,\pi]$, $g(0)=g(\pi)=0$; there exists a unique classical solution of (HC) (1), (2), (3).

Proof: Considering the real and imaginary parts of $u$ and $g$, we can assume from the very beginning that $g$ is real (and so it is $u_t$).
We shall prove uniqueness first: let $u : [0,\pi]\times[0,+\infty[ \longrightarrow \mathbb{R}$ be a classical solution. Fix $0<T$. Consider the restriction $u : [0,\pi]\times[0,T] \longrightarrow \mathbb{R}$
We know, by the previous theorem, that

$$u(x,t) \leq \|g\|_\infty, \quad x\in[0,\pi], \ 0\leq t\leq T$$

Applying the theorem again to $-g$ (solution is $-u$) we get

$$-u(x,t) \leq \|g\|_\infty, \quad x\in[0,\pi], \ 0\leq t\leq T$$

so $$|u(x,t)| \leq \|g\|_\infty, \quad x\in[0,\pi], \ 0\leq t\leq T$$

from which uniqueness follows (if $g\equiv 0$, then $u\equiv 0$; if $u_1, u_2$ are two solutions to the problem with $g$, $u_1-u_2$ is a solution to the problem with $0$, so $u_1=u_2$).

Existence: extend $g$ to an odd function on $[-\pi,\pi]$ and then to a $2\pi$-periodic function $g\in\mathcal{C}$. Let $g\sim\sum_{n\in\mathbb{Z}} b_n e_n$.
Let $\varphi_m \sim \sum_{n\in\mathbb{Z}} a_{mn} e_n$ be an approximate identity. We know

$$|a_{mn}|\leq 1, \ \forall m\in\mathbb{N}, \forall n\in\mathbb{Z}$$

$$a_{mn} \xrightarrow{m} 1, \ \forall n\in\mathbb{Z}$$

Solve the problem for $g_m := \varphi_m * g \in \mathcal{P}$ and get $(F_m)_t$, $t>0$

$$(F_m)_t = F_{(u_m)_t} \sim \sum_{n\in\mathbb{Z}} a_{mn}(t)\, e_n, \quad t>0, m\in\mathbb{N}$$

Solve the problem for $g$ and get $(F_t)_{t>0}$

$$F_t = F_{u_t} \sim \sum_{n\in\mathbb{Z}} a_n(t) e_n, \quad t>0. \qquad g_m \sim \sum_{n\in\mathbb{Z}} a_{mn} b_n . e_n$$

$$a_{mn}(t) = a_{mn}.b_n.\exp(-n^2t), \quad m\in\mathbb{N}, n\in\mathbb{Z}, t>0$$
$$a_n(t) = b_n \exp(-n^2t), \quad n\in\mathbb{Z}, t>0$$

$$\boxed{G \in \mathcal{P}' \text{ even (odd)} \Rightarrow F_t \text{ even (odd)}, \forall t > 0}$$

⊠ $G \sim \sum_{\mathbb{Z}} b_n e_n$ even $\equiv G = \tilde{G} \equiv G(u) = G(\tilde{u}), \forall u \in \mathcal{P}$

In particular, $G(e_{-n}) = G(\tilde{e}_{-n}), \forall n$

$\tilde{e}_{-n}(x) = e_{-n}(-x) = \exp(inx) = e_n(x) \Rightarrow \tilde{e}_{-n} = e_n$

$G(e_{-n}) = G(e_n), \forall n \Rightarrow b_n = b_{-n}, \forall n$

Now, $F_t \sim \sum_{\mathbb{Z}} a_n(t) e_n, \forall t > 0$

$a_n(t) = b_n \cdot \exp(-n^2 t), \ t > 0, n \in \mathbb{Z}$

$a_{-n}(t) = b_{-n} \exp(-n^2 t) = a_n(t), \ t > 0, n \in \mathbb{Z}$

$\Rightarrow u_t, F_t$ is even.

Analogously for the odd case. □

$$\boxed{G \in \mathcal{P}' \text{ real} \Rightarrow F_t \text{ real}, \forall t > 0}$$

⊠ $G = G^*$

$G(u) = G^*(u) = [G(u^*)]^*$. In particular, $G(e_{-n}) = [G(e^*_{-n})]^* =$
$= (G(e_n))^*$, hence $b_n = b^*_{-n}, \forall n \in \mathbb{Z}$.

The same is true for $a_n(t)$:

$a_n(t) = b_n \exp(-n^2 t) = b^*_{-n} \exp(-n^2 t) = (b_{-n} \exp(-n^2 t))^* =$
$= (a_{-n}(t))^* \Rightarrow F_t$ real, $\forall t > 0$. □

$$\boxed{G \in \mathcal{P}',\ G = F_g,\ g \in \mathcal{P} \Rightarrow (u_t = F_{u_t} = F_t)_{t>0} \xrightarrow[t \to 0]{\mathcal{P}} G}$$

⊠ $F(t) \sim \sum_{n \in \mathbb{Z}} a_n(t) e_n$

$a_n(t) = b_n \cdot \exp(-n^2 t), \ n \in \mathbb{Z}, t > 0$

$|a_n(t)| = |b_n| \cdot \exp(-n^2 t) \leq |b_n|, \ \forall n \in \mathbb{Z}, \forall t > 0$

Now, $\forall r > 0 \ \exists c(r) > 0$ such that $|b_n| \leq |n|^{-r} \cdot c(r), \ \forall n \in \mathbb{Z}, n \neq 0$

We get $|a_n(t)| \leq |n|^{-r} \cdot c(r), \ \forall n \in \mathbb{Z}, n \neq 0, \forall t > 0$

and then $F_t \xrightarrow{\mathcal{P}} G$. □

$(\varphi_m)_1^\infty \subset \mathcal{C}$ an approximate identity. Let $\varphi_m \sim \sum_{n\in\mathbb{Z}} a_{mn} e_n$, $\forall m \in \mathbb{N}$

Then $|a_{mn}| \leq 1$, $\forall m \in \mathbb{N}$, $\forall n \in \mathbb{Z}$ and

$$a_{mn} \xrightarrow[m\to\infty]{} 1, \ \forall n \in \mathbb{Z}$$

☒ We know that $\varphi_m * u \xrightarrow[m\to\infty]{\|\cdot\|_\infty} u$, $\forall u \in \mathcal{C}$. In particular

$$\varphi_m * e_n \xrightarrow[m\to\infty]{\|\cdot\|_\infty} e_n, \ \forall n \in \mathbb{Z}$$

$$(\varphi_m * e_n)(x) = \frac{1}{2\pi}\int_0^{2\pi} e_n(x-y)\,\varphi_m(y)\,dy$$

$$e_n(x-y) = \exp(in(x-y)) = \exp(inx)\cdot\exp(-iny)$$

$$(\varphi_m * e_n)(x) = \frac{1}{2\pi}\exp(inx)\int_0^{2\pi}\exp(-iny)\,\varphi_m(y)\,dy = \exp(inx)\cdot\langle \underline{e}_n, \varphi_m\rangle =$$
$$= \exp(inx)\cdot a_{mn}$$

$(\varphi_m * e_n)(x) \xrightarrow[m\to\infty]{} e_n(x) = \exp(inx)$, $\forall x \in \mathbb{R}$ (the convergence is uniform)

In particular, $(\varphi_m * e_n)(0) \xrightarrow[m\to\infty]{} e_n(0) = 1$,

It follows that $\exp(in0)\cdot a_{mn} = a_{mn} \xrightarrow[m\to\infty]{} 1$.

$$a_{mn} = \langle \underline{e}_n, \varphi_m\rangle = \frac{1}{2\pi}\int_0^{2\pi} \underline{e}_n(y)\,\varphi_m(y)\,dy$$

$$|a_{mn}| \leq \frac{1}{2\pi}\int_0^{2\pi}\varphi_m(y)\,dy = 1, \ \forall m \in \mathbb{N}, \forall n \in \mathbb{Z}. \ \square$$

Theorem

Let $G \in \mathcal{P}'$, and consider the problem : find $(F_t)_{t>0}$ in $\mathcal{P}'$ such that

$$(H) \begin{cases} \left.\dfrac{d}{dt} F_t\right|_{t=s} = D^2 F_t \\ F_t \xrightarrow[t\to 0]{\mathcal{P}'} G \end{cases}$$

Then $(H)$ has one and only one solution $(F_t)_{t>0} \subset \mathcal{P}'$. Moreover $F_t = F_{u_t}$, for some $u_t \in \mathcal{P}$, $t>0$ and the function $u(x,t) = u_t(x)$, $x \in \mathbb{R}$, $t>0$ is also infinitely differentiable as a function of $t$, in $]0,+\infty[$

Proof: We shall prove uniqueness first :

Assume $(F_t)_{t>0}$ is a family in $\mathcal{P}'$ such that $(H)$ is satisfied.

Let $G \sim \sum_{\mathbb{Z}} b_n e_n$, $F_t \sim \sum_{n\in\mathbb{Z}} a_n(t) e_n$, $t>0$

Consider $\dfrac{F_{t+\Delta t} - F_t}{\Delta t} \in \mathcal{P}'$, where $t$, $t+\Delta t$ are both $>0$.

We know that $\dfrac{F_{t+\Delta t} - F_t}{\Delta t} \xrightarrow[\Delta t\to 0]{\mathcal{P}'} D^2 F_t$

In particular,

$$\left\langle e_{-n}, \frac{F_{t+\Delta t} - F_t}{\Delta t} \right\rangle \longrightarrow \langle e_{-n}, D^2 F_t\rangle = \langle D^2 e_{-n}, F_t\rangle =$$

$$= -n^2 \langle e_{-n}, F_t\rangle = -n^2 a_n(t)$$

$$\left\langle e_{-n}, \frac{F_{t+\Delta t} - F_t}{\Delta t} \right\rangle = \frac{a_n(t+\Delta t) - a_n(t)}{\Delta t} \xrightarrow[\Delta t\to 0]{} -n^2 a_n(t).$$

It follows that

$$\exists \boxed{a_n'(t) = -n^2 a_n(t), \quad \forall n\in\mathbb{Z}, \forall t>0}$$

Moreover $\langle e_{-n}, F_t\rangle \xrightarrow[t\to 0]{} \langle e_{-n}, G\rangle$ ; $\forall n\in\mathbb{Z}$

so $\boxed{a_n(t) \xrightarrow[t\to 0]{} b_n}$

There exists one and only one solution $\boxed{a_n(t) = b_n \cdot \exp(-n^2 t), \ n\in\mathbb{Z}, t>0}$

This proves uniqueness.

In order to prove existence consider

$$\boxed{a_n(t) = b_n \cdot \exp(-n^2 t),\ n\in\mathbb{Z},\ t>0}$$

$(b_n)$ is the sequence of Fourier coefficients of $G\in\mathcal{P}'$, hence $\exists r>0, \exists c>0$ such that $|b_n| \leq c|n|^r,\ \forall n\in\mathbb{Z}, n\neq 0$

Fix $t>0$. Note that $|a_n(t)| \leq |b_n| \leq c|n|^r,\ \forall n\in\mathbb{Z}, n\neq 0$

so $(a_n(t))$ is the sequence of Fourier coefficients of a certain $F_t\in\mathcal{P}'$.

We can say much more: Fix K.

$$|a_n(t)| = |b_n| \cdot \exp(-n^2 t) = |b_n| \cdot \frac{1}{\exp(n^2 t)} \leq$$

$$\leq |b_n| \cdot \frac{p!}{(n^2 t)^p} \leq c|n|^r \cdot \frac{p!}{|n|^{2p} \cdot t^p} = \frac{c \cdot p!}{t^p} |n|^{r-2p}$$

It is enough to choose $p$ such that $r-2p<-K$ to get that $(a_n(t))_{n\in\mathbb{Z}}$ is the sequence of Fourier coefficients of a distribution $F_t\in\mathcal{P}$, i.e., there exists a function $u_t\in\mathcal{P}$ such that $F_t = F_{u_t}$ $(t>0)$.

Define $u(x,t) = u_t(x),\ \forall x\in\mathbb{R},\ \forall t>0$

$$\boxed{F_t \sim \sum_{n\in\mathbb{Z}} a_n(t) e_n}$$

We want to prove that $F_t \xrightarrow{\mathcal{P}'} G$ as $t\to 0$. By ( ) it is enough to prove that

$$a_n(t) \xrightarrow[t\to 0]{} b_n$$

$$|a_n(t)| \leq c|n|^r,\ t>0,\ n\in\mathbb{Z}, n\neq 0$$

for some $c>0, r>0$. Observe that

$$|a_n(t)| \leq |b_n| \leq c|n|^r,\ \forall n\in\mathbb{Z}, n\neq 0,$$

and $a_n(t) = b_n \exp(-n^2 t) \xrightarrow[t\to 0]{} b_n$

so both conditions are satisfied and $F_t \xrightarrow[t\to 0]{\mathcal{P}'} G$.

It remains to prove that, for some $t>0$,

$$\frac{F_{t+\Delta t} - F_t}{\Delta t} \xrightarrow{\mathcal{P}'} D^2 F_t \qquad (*)$$

Observe that $$\frac{F_{t+\Delta t} - F_t}{\Delta t} \sim \sum_{n\in\mathbb{Z}} \frac{a_n(t+\Delta t) - a_n(t)}{\Delta t} e_n$$

and $$D^2 F_t \sim \sum_{n\in\mathbb{Z}} -n^2 a_n(t) e_n = \sum_{n\in\mathbb{Z}} a_n'(t) e_n$$

(We know that $-n^2 a_n(t) = a_n'(t)$, $t>0$, $n\in\mathbb{Z}$). Then

$$\frac{a_n(t+\Delta t)-a_n(t)}{\Delta t} \xrightarrow[\Delta t\to 0]{} a_n'(t) = -n^2 a_n(t)$$

In order to prove $(*)$, we shall use again ( ):

$$\frac{a_n(t+\Delta t)-a_n(t)}{\Delta t} = a_n'(t+\theta_n\Delta t) \quad \text{, for some } 0<\theta_n<1.$$

It is enough to prove that $\exists c>0$, $\exists k>0$ such that

$$|a_n'(t+\theta_n\Delta t)| \leq c|n|^k, \quad \forall n\in\mathbb{Z}, n\neq 0, \quad |\Delta t|<\delta$$

$a_n'(s) = -n^2 b_n \exp(-n^2 s)$, $n\in\mathbb{Z}$, $s>0$

$$|a_n'(s)| \leq |n|^2 . c|n|^r \frac{p!}{(n^2 s)^p} \quad ;$$

$$|a_n'(s)| \leq |n|^{r+2-2p} c . p! \frac{1}{s^p}$$

Then,

$$|a_n'(t+\theta_n\Delta t)| \leq |n|^{r+2-2p} . c . p! \frac{1}{(t-\Delta t)^p} \leq |n|^{r+2-2p} c . p! \frac{1}{(t-\delta)^p}$$

It is enough to take $p$ such that $r+2-2p$. $\square$

## The heat equation

The classical heat equation appears as

$$\boxed{\frac{\partial}{\partial t} u(x,t) = K \frac{\partial^2}{\partial x^2} u(x,t) \quad , \quad x \in \left]0,\pi\right[, t>0} \qquad (1)$$

Where $u : [0,\pi] \times [0,+\infty[ \longrightarrow \mathbb{R}$ is a continuous function such that the appropriate partial derivatives exist.

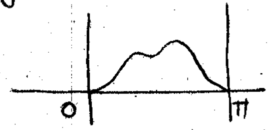
We want to find $u$ such that it satisfies (1)

and, moreover, given a continuous function $g$ on $[0,\pi]$,

$$\boxed{u(x,0) = g(x), \ \forall x \in [0,\pi]} \qquad (2)$$

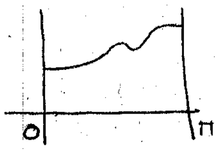
Even more, we add one of the two following conditions

$$\boxed{u(0,t) = u(\pi,t) = 0, \ \forall t \geq 0} \qquad (3)$$




or

$$\boxed{\frac{\partial}{\partial x} u(0,t) = \frac{\partial}{\partial x} u(\pi,t) = 0, \quad \forall t>0} \qquad (3')$$




Now, we can write a single set of conditions just by extending our function $g$ and the solution to $[-\pi,\pi]$, in case (3) using an odd extension, in case (3') using an even extension.

So finally write

$$\boxed{\begin{array}{l} \frac{\partial}{\partial t} u(x,t) = \frac{\partial^2}{\partial x^2} u(x,t) \quad , \ x \in \mathbb{R}, t>0 \\ u(x,0) = g(x), \ x \in \mathbb{R} \end{array}}$$

It is more general (and simple) to look at a corresponding problem formulated in terms of distributions (periodic distributions)

Find a family of distributions $(F_t)_{t>0}$ (they correspond to our former functions $u_t(x) = u(x,t)$), $F_t \in \mathcal{P}'$, such that

$$\boxed{\begin{array}{l} \left.\frac{d}{dt} F_t\right|_{t=s} = D^2 F_s \, , \quad \forall s>0 \\ F_t \xrightarrow[t\to 0]{\mathcal{P}'} G \quad , \ \text{for some } G \in \mathcal{P}' \end{array}} \qquad (H)$$

We need to attribute a meaning to $\frac{d}{dt}F_t$.

Consider $\frac{F_t - F_s}{t-s} \in \mathcal{P}'$

where $t>0, s>0$

Suppose that $\frac{F_t - F_s}{t-s} \xrightarrow{\mathcal{P}'}$ some element in $\mathcal{P}'$

We denote this limit as $\left.\frac{d}{dt}F_t\right|_{t=s} \in \mathcal{P}'$

Then we have the following

$\boxed{\begin{array}{l}\text{Theorem: } \forall G \in \mathcal{P}' \text{ there exists a unique family } (F_t)_{t>0} \\ \text{of elements in } \mathcal{P}' \text{ such that it satisfies (H).} \\ \text{Moreover, } F_t \text{ is in fact a distribution associated to a function} \\ u_t,\ t>0,\ u_t \in \mathcal{P},\ \forall t>0; \text{ And } (x,t) \longrightarrow u_t(x) = u(x,t) \text{ is} \\ \text{infinitely differentiable also with respect to } t.\end{array}}$

Proof: We shall prove uniqueness first: $G \sim \sum b_n e_n$

Suppose $F_t \sim \sum_{-\infty}^{\infty} a_n(t) e_n$, $t>0$ ; where $a_n(t) = F_t(e_{-n})$

Then $\frac{F_t - F_s}{t-s} \xrightarrow[t\to s]{\mathcal{P}'} \left.\frac{d}{dt}F_t\right|_{t=s} = D^2 F_s$

so, in particular

$$\left\langle e_{-n}, \frac{F_t - F_s}{t-s} \right\rangle \xrightarrow[t\to s]{} \langle e_{-n}, D^2 F_s \rangle = F_s(D^2 e_{-n}) = F_s(-n^2 e_{-n}) =$$

$$= -n^2 F_s(e_{-n}) = -n^2 a_n(s)$$

Observe that $\left\langle e_{-n}, \frac{F_t - F_s}{t-s} \right\rangle = \frac{1}{t-s}\left[a_n(t) - a_n(s)\right]$

Then we get that $a_n(t)$ has a derivative and

$$\boxed{Da_n(t) = -n^2 a_n(t)}$$

Moreover $F_t(e_{-n}) \xrightarrow[t\to 0]{} G(e_{-n})$, so $\boxed{a_n(t) \xrightarrow[t\to 0]{} b_n}$

This two conditions are satisfied by only one function

$$a_n(t) = b_n \exp(-n^2 t), \quad n \in \mathbb{Z}, \ t > 0 \qquad [*]$$

and we proved uniqueness.

To prove existence use the previous calculation:

We know that $(b_n)$ is a slow growth sequence. Then

$$\exists r > 0, \exists c > 0, \quad |b_n| \leq c|n|^r, \ \forall n \neq 0$$

Fix $t > 0$. From $[*]$ we get that $(a_n(t))_{n \in \mathbb{Z}}$ is a sequence of rapid decrease. It follows that it is the sequence of Fourier coefficients of a function $u_t \in \mathcal{P}$.

We shall prove that $(F_t)_{t>0}$ satisfies $(H)$.

First, $|a_n(t)| = |b_n| \cdot |\exp(-n^2 t)| \leq |b_n| \leq c \cdot |n|^r, \ \forall n \neq 0$

Using the Theorem on convergence proved in the previous section, we get $F_t \xrightarrow[t \to 0]{\mathcal{P}'}$ some element in $\mathcal{P}'$ with Fourier coefficients $\lim_{t \to 0} a_n(t) = b_n$

This element is $G$.

$$\frac{F_t - F_s}{t - s} \sim \sum_{\mathbb{Z}} \frac{a_n(t) - a_n(s)}{t - s} e_n$$

We shall prove that, as $t \to s$, all coefficients in the last expression are «uniformly bounded» for slow growth. If this is so,

$$\frac{F_t - F_s}{t - s} \xrightarrow[t \to s]{\mathcal{P}'} M \in \mathcal{P}', \quad M \sim \sum_{\mathbb{Z}} a_n'(s) e_n$$

$$a_n'(s) = -n^2 b_n \exp(-n^2 s) = -n^2 a_n(s)$$

$$M \sim \sum_{\mathbb{Z}} -n^2 a_n(s) e_n$$

$$\text{so} \quad M = D^2 F_s$$

Now, $\dfrac{a_n(t) - a_n(s)}{t - s} = a_n'(p)$, for some $p$ between $t$ and $s$.

$$|a_n'(p)| = -|n|^2 |a_n(p)| = -|n|^2 |b_n| \exp(-n^2 p)$$

$$\leq -|n|^2 \cdot c \, |n|^{+r} \exp(-n^2 p) \qquad \square$$

$$u_t(x) = \sum_{\mathbb{Z}} a_n(t) e_n(x) = u(x,t)$$

Fix $x \in \mathbb{R}$. The kind of convergence of this serie of functions and theorems on differentiation of series say that $u(x, \cdot)$ is infinitely differentiable. $\square$

## An application to holomorphic functions on a disc

Consider an holomorphic ($\equiv$ derivable) function $f: D \to \mathbb{C}$, where $D$ is a disk (an open disc) in $\mathbb{C}$, say

$$D := \{ z \in \mathbb{C} : |z - z_0| < r \}$$

for some $z_0 \in \mathbb{C}$, $0 < r$. A simple change of variable allow us to consider $D$ the open unit disc

$$D := \{ z \in \mathbb{C} : |z| < 1 \}$$

We shall discuss values of $f$ on $D$ versus values on the boundary (or assymptotic behaviour of $f$ approaching the boundary).
Define a function of two variables

$$g(r,\theta) := f(re^{i\theta}) \quad ; \quad 0 \leq r < 1, \ \theta \in \mathbb{R}$$

Alternatively, we can think of a family $(g_r)_{r \in [0,1[}$ of $2\pi$-periodic functions on $\mathbb{R}$, where

$$g_r(\theta) = f(r e^{i\theta}) = g(r,\theta), \quad 0 \leq r < 1, \ \theta \in \mathbb{R}$$

Obviously, this function has partial derivatives (and $g_0$ is a constant function). Partial derivatives of $g$ are not arbitrary (think of the Cauchy-Riemann equations). Precisely

$$\begin{cases} \dfrac{\partial g}{\partial r} = e^{i\theta} f'(re^{i\theta}) \\ \dfrac{\partial g}{\partial \theta} = r.i.e^{i\theta} f'(re^{i\theta}) \end{cases}$$

Hence $\boxed{\dfrac{\partial g}{\partial \theta} = ri \dfrac{\partial g}{\partial r}}$

Given $0 \leq r < 1$, $g_r$ is a function in $\mathcal{P}$. Therefore, it is the sum of its Fourier series

$$g_r(\theta) = \sum_{n \in \mathbb{Z}} a_n(r).e^{in\theta}, \quad 0 \leq r < 1, \ \theta \in \mathbb{R}$$

## Laplace equation

Let's consider the following problem

Find a function $u: D \to \mathbb{R}$ such that $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$ (Laplace equation) where $D \equiv \{(x,y) : x^2+y^2 \leq 1\} = B(0;1)$ and such that $u(x,y) = g(x,y)$, $x^2+y^2=1$, where $g$ is a given function on the boundary of $D$ (Dirichlet's problem).

Change the problem to polar coordinates:

$$\text{Solve} \left[ \begin{array}{ll} r^2 \frac{\partial^2 u}{\partial r^2} + r\frac{\partial u}{\partial r} + \frac{\partial^2 u}{\partial \theta^2} = 0, & 0 \leq r < 1, \ \theta \in \mathbb{R} \\ u(1,\theta) = g(\theta), \ \theta \in \mathbb{R} & \end{array} \right] \quad (D) \qquad (L)$$

where $u$ and $g$ are $2\pi$-periodic

**Theorem**: Given $F \in \mathcal{P}'$, there exists a unique function $u(r,\theta)$ defined and infinitely differentiable in $0 \leq r < 1$, $\theta \in \mathbb{R}$ such that $u_r \xrightarrow[r\to 1]{\mathcal{P}'} F$ and such that satisfies $(L)$.

Moreover, if $F = F_g$, for some $g \in \mathcal{E}$, then $u_r \xrightarrow[r \to 1]{\|\cdot\|_\infty} g$ and, in general,

$$u_r = F * P_r$$

where $P_r$ is the Poisson Kernel

$$P_r \in \mathcal{P}, \quad P_r(\theta) = \sum_{n\in\mathbb{Z}} r^{|n|} e^{in\theta}$$

Proof: Let $F \sim \sum b_n e_n$

Assume first that we have already a family $F_r = u_r$ in $\mathcal{P}'$ which is a solution of $(D)$.

$u_r = F_r \sim \sum_{n\in\mathbb{Z}} a_n(r) e_n$ (the variable is $\theta$)

Then $\quad r^2 D^2 a_n + r D a_n - n^2 a_n = 0 \quad , 0 < r < 1$

$\langle e_{-n}, u_r \rangle \xrightarrow[r\to 1]{} \langle e_n, F \rangle = b_n$, then $a_n(r) \xrightarrow[r \to 1]{} b_n$, $n \in \mathbb{Z}$

Moreover $\quad u_0(\theta) \sim \sum_{n\in\mathbb{Z}} a_n(0) e_n(\theta)$, $\forall \theta$, and $u_0(\theta)$ is a constant.

Then $a_n(0) = 0, n \neq 0$

We shall consider a solution of the form $a_n(r) = b_n \cdot r^c$, where $c = c(n)$ is some constant. Then

$$(c-1) r^2 b_n c\, r^{c-2} + r\, b_n c\, r^{c-1} - n^2 b_n r^c =$$
$$= b_n [r^c] \left( c(c-1) + c - n^2 \right] = 0$$

and we get $c^2 - n^2 = 0$.

$a_n(r) = b_n . r^{c(n)}$

$a_n(0) = 0 \Rightarrow c(n) = |n|$, $n \neq 0$

so $a_n(r) = b_n . r^{|n|}$ , $a_0(r) = b_0$

$$F_r \sim \sum_{n\in\mathbb{Z}} b_n r^{|n|} \exp(in\theta)$$

$$u(r,\theta) \sim \sum_{n\in\mathbb{Z}} b_n . r^{|n|} \exp(in\theta) = F * P_r$$

where $P_r$ is a distribution whose Fourier coefficients are $(r^{|n|})_{n\in\mathbb{Z}}$

Observe that in case $0 \leq r < 1$, this sequence gives a distribution $P_r \in \mathcal{P}$ (it is obviously of rapid decrease)

$$P_r(\theta) = \sum_{n\in\mathbb{Z}} r^{|n|} . e^{in\theta} \quad , \quad 0 \leq r < 1$$

$(P_r)$ is an approximate identity for $r \to 1$ and it is called the <u>Poisson Kernel.</u>

This proves uniqueness.

In order to prove existence, let's define

$$U_r = F * P_r \quad , \quad 0 \leq r < 1$$

Observe that $(P_r)_{0\leq r<1}$ is an approximate identity, for $r\to 1$ hence

$$U_r \xrightarrow[r\to 1]{\mathcal{P}'} F$$

and in case $F = F_g$, $g \in \mathcal{C}$, $U_r \xrightarrow{\|\cdot\|_\infty} F$

We shall prove that $U_r(\theta) = u(r,\theta)$, $0 \leq r < 1$, $\theta \in \mathbb{R}$ satisfies $(L)$ in this domain:

Take $0 < R < 1$. We shall prove that $u$ is infinitely differentiable in $B(0;R)$ and satisfies $(L)$:

Let $r < R$, $s = r/R < 1$

$$U_r = F * P_r = F * (P_R * P_s) = (F * P_R) * P_s = U_R * P_s$$

$$u(r,\theta) = \frac{1}{2\pi} \int_0^{2\pi} U_R(\alpha) \, P_s(\theta - \alpha) d\alpha \quad , \quad 0 \leq r < R$$

It is enough to differentiate under the integral to prove that $u$ is infinitely differentiable and satisfies $(L)$.

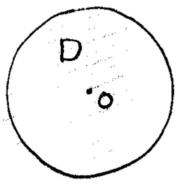
## Laplace equation

$$\boxed{\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}=0} \qquad [L] \qquad (\text{Laplace equation})$$

A function $u$ satisfying $[L]$ is called <u>harmonic</u>

<u>Dirichlet's problem</u>

Find $u:\overline{D}(\{z:|z|\leq 1\})\longrightarrow \mathbb{C}$ continuous on $\overline{D}$, harmonic on $D$, and equal on the boundary to a given function $g$




The problem is better treated if we consider polar coordinates; then $(L)$ appears as

$$\begin{cases} r^2\dfrac{\partial^2 u}{\partial r^2}+r\dfrac{\partial u}{\partial r}+\dfrac{\partial^2 u}{\partial \theta^2}=0, & 0\leq r<1,\ \theta\in\mathbb{R} \qquad (L)\\ u(1,\theta)=g(\theta), & \theta\in\mathbb{R}\end{cases}$$

We shall solve the problem formally. Later we shall justify the derivation: let's call

$u_r:\mathbb{R}\to\mathbb{C}$

$u_r(\theta)=u(r,\theta)$, $0\leq r<1$, $\theta\in\mathbb{R}$, a periodic function

$u_r\sim\sum_{n\in\mathbb{Z}} a_n(r)\,e_n$, $0\leq r<1$ (observe that $u_0$ is a constant function)

Then we find, for $0<r<1$, $n\in\mathbb{Z}$, $g\sim\sum_{\mathbb{Z}} b_n e_n$,

$$\begin{cases} r^2D^2a_n(r)+rDa_n(r)-n^2a_n(r)=0\\ a_n(1)=b_n,\ n\in\mathbb{Z}\\ a_n(0)=0,\ n\neq 0\end{cases}$$

Consider a solution $a_n(r)=b_n.r^c$, for some constant $c=c(n)$.

Then

$$c(c-1)+c-n^2=0$$

$$c^2-n^2=0$$

We want $a_n(0)=0$, $n\neq 0$, then $c=|n|$. We get

$$a_n(r)=b_n.r^{|n|}$$

$$u(r,\theta)=u_r(\theta)=\sum_{\mathbb{Z}}a_n(r)e_n(\theta)=\sum_{n\in\mathbb{Z}}b_n r^{|n|}e^{in\theta}$$

It follows that $u_r$ is formally the convolution of $g$ and $P_r(\theta)$, where $\boxed{P_r(\theta)=\sum_{n\in\mathbb{Z}}r^{|n|}e_n(\theta),\ 0\leq r<1,\ \theta\in\mathbb{R}}$

(Poisson Kernel)

$$\boxed{u_r=g*P_r}$$

$$P_r(\theta) = 1 + \sum_1^{+\infty} r^n e^{in\theta} + \sum_1^{\infty} r^n e^{-in\theta} = 1 + \frac{re^{i\theta}}{1-re^{i\theta}} + \frac{re^{-i\theta}}{1-re^{-i\theta}} =$$

$$= \frac{(1-re^{i\theta})(1-re^{-i\theta}) + re^{i\theta}[1-re^{-i\theta}] + re^{-i\theta}[1-re^{i\theta}]}{|1-re^{i\theta}|^2} = \frac{1-r^2}{|1-re^{i\theta}|^2} =$$

$$= \frac{1-r^2}{1-2r\cos\theta + r^2}$$

$(P_r)_{0\le r<1}$ is an approximate identity as $r \to 1$

☒ $$\frac{1}{2\pi}\int_0^{2\pi} P_r(\theta)d\theta = 1 + \sum_1^{\infty} \frac{r^n}{2\pi}\int_0^{2\pi} e^{in\theta}d\theta + \sum_1^{\infty} \frac{r^n}{2\pi}\int_0^{2\pi} e^{-in\theta}d\theta = 1$$

$P_r(\theta) \geq 0$, $0\le r<1$, $\theta \in \mathbb{R}$

$$\lim_{r\to 1} \int_{\delta}^{2\pi-\delta} P_r(\theta)d\theta = 0 \quad \square$$

**Theorem** Let $F$ be an element in $\mathcal{P}'$, say $F \sim \sum_{\mathbb{Z}} b_n e_n$
Then there exists a unique $u(r,\theta)$, $0\le r<1$, $\theta\in\mathbb{R}$
which is infinitely differentiable in $D$ (then $u_r(\theta) = u(r,\theta)$ is in $\mathcal{P}$ for $0\le r<1$) which satisfies the Laplace equation in $D$ ($\equiv$ it is harmonic) and $u_r \xrightarrow[r\to 1]{\mathcal{P}'} F$, and

$$u_r = F * P_r \quad , 0\le r<1$$

In case $F = F_g$, $g\in\mathcal{C}$, then $u_r \xrightarrow[r\to 1]{\|\,\|_\infty} g$

Proof: We already proved uniqueness.
Assume now $u_r \equiv F * P_r$. As $(P_r)_{0\le r<1}$ is an approximate identity for $r\to 1$ we know that

$$u_r = F * P_r \xrightarrow[r\to 1]{\mathcal{P}'} F$$

We also know $u_r \in \mathcal{P}$, $0\le r<1$. Assume $F = F_g$, $g\in\mathcal{C}$:

$$u_r(\theta) = u(r,\theta) = \frac{1}{2\pi}\int_0^{2\pi} P_r(\theta - z)\, g(z)\,dz$$

compute $\partial u/\partial r$ and $\partial u/\partial\theta$ and the succesive derivatives to proof that $u$ is infinitely differentiable in $D$
Assume now $F\in\mathcal{P}'$.

$$0 \le r < s < 1 \Rightarrow P_r * P_s = P_{rs}$$

Choose $0<R<1$. It is enough to prove that $u$ is infinitely differentiable in $B(0;R)$ and satisfies (L) there: Let $r<R$, $s=r/R<1$

$$u_r = F*P_r = F*(P_R*P_s) = (F*P_R)*P_s = u_R*P_s$$

$$P_R \in \mathcal{P} \Rightarrow u_R \in \mathcal{P}$$

$$u(r,\theta) = \frac{1}{2\pi}\int_0^{2\pi} P\left(\frac{r}{R}, \theta - z\right) u_R(z)\,dz \quad , 0 \le r < R$$

Differentiate to prove that $u(r,\theta)$ is infinitely differentiable. and satisfies (L). $\square$

## Remark

How to prove that $a_n(r) = a_n . r^n$ for some constant $a_n$, and $a_n = 0$ for $n<0$

We start from

$$\boxed{a_n(r) . \frac{n}{r} = a_n'(r), \quad 0<r<1, \qquad a_n \text{ continuous on } [0,1[}$$

$a_n$ is a complex-valued function defined on $[0,1[$

Observe that its real and imaginary parts satisfied the same equation, so consider a real-valued function

$$\boxed{\begin{array}{l} u: [0,1[ \longrightarrow \mathbb{R} \\ \text{continuous on } [0,1[ \\ \text{differentiable on } ]0,1[ \\ u(r) . \frac{n}{r} = u'(r), \quad 0<r<1 \end{array}}$$

Assume first $u(0)=0$

Let $S = \{ r \in [0,1[, \ u(r)=0 \}$. Let $s = \sup S \in [0,1]$

We shall prove that $S$ is dense in $[0,s]$. Assume this is not the case. We can find then an open interval (non-empty) (relative to $[0,s]$) where $u$ is different from $0$ (If $s=0$ there is nothing to prove, and then $u(r) \neq 0$, $r \in ]0,1[$).

Choose this interval to be maximal, say $]a,b[$. Then $u(a)=u(b)=0$. By the mean-value theorem, there exists $c \in ]a,b[$ such that $u'(c) = u(c) . \frac{n}{c} = 0$, so $u(c)=0$, a contradiction.

Now, $u$ is continuous on $[0,1[$, hence $u(x)=0$, $\forall x \in [0,s]$.

In case $s=1$; we get $u \equiv 0$

Assume $s<1$. It follows that $u(r) \neq 0$ in $]s,1[$

$\frac{u'(r)}{u(r)} = \frac{n}{r}$, $\forall r \in ]s,1[$, then $u(r) = a . r^n$, $r \in ]s,1[$, for some constant $a$. As $u$ is continuous on $[s,1[$, we get $u(s) = a s^n = 0$.

If $s>0$, we get a contradiction.

So, if $u(0)=0$, $u(r) = a r^n$, $r \in [0,1[$ for some constant $a$.

If $u(0) \neq 0$, then $u(r) \neq 0$, $\forall r \in ]0,1[$ (otherwise we can repeat the argument in $[r_0,1[$, where $u(r_0)=0$, and reach again a contradiction). It follows that $u(r) = a r^n$, $r \in [0,1[$ and we get a contradiction if $a \neq 0$, $n>0$.

Summarizing: $a_n(r) = a_n r^n$, $r \in [0,1[$, $n>0$, $a_n(r) = 0$, $n<0$

$a_0(r) = a_0$ (a constant).

where $a_n(r) = \frac{1}{2\pi}\int_0^{2\pi} g_r(\theta)\, e^{-in\theta}\, d\theta$ , $n\in\mathbb{Z}, r\in[0,1[$

then $$a_n(r) = \frac{1}{2\pi}\int_0^{2\pi} f(re^{i\theta})\, e^{-in\theta}\, d\theta = \frac{1}{2\pi}\int_0^{2\pi} g(r,\theta)\, e^{-in\theta}\, d\theta$$

and it is a continuous function on $[0,1[$, differentiable in $]0,1[$

and $$a_n'(r) = \frac{1}{2\pi}\int_0^{2\pi} \frac{\partial}{\partial r} g(r,\theta)\, e^{-in\theta}\, d\theta = \frac{1}{2\pi}\int_0^{2\pi} \frac{1}{ri}\frac{\partial g}{\partial\theta}(r,\theta)\, e^{-in\theta}\, d\theta =$$

(Use integration by parts : $u = e^{-in\theta}$, $dv = \frac{\partial g}{\partial\theta}$

$du = -in\, e^{-in\theta}\, d\theta$ , $v = g$ )

$$= \frac{1}{r2\pi i}\left[ g(r,\theta)\, e^{-in\theta}\Big|_{\theta=0}^{\theta=2\pi} + in\int_0^{2\pi} g(r,\theta).e^{-in\theta}\, d\theta\right] =$$

$$= \boxed{\frac{n}{r}\, a_n(r) = a_n'(r)}$$

Observe that $a_0'(r) = 0$ , $r\in]0,1[$ , and $a_0$ continuous in $[0,1[$, hence $a_0$ is constant in $[0,1[$.

We have also $g_0(\theta) = \sum_{n\in\mathbb{Z}} a_n(0)\, e^{in\theta}$ , a constant function. Then

$a_n(0) = 0,\ n\neq 0$

It follows that

$$\boxed{\begin{array}{l} a_n(r) = a_n . r^n \text{ for some constant } a_n, \\ \text{and } a_n = 0 \text{, for } n<0 \end{array}}$$

So $$g_r(\theta) = g(r,\theta) = f(r.e^{i\theta}) = \sum_{n\geq 0} a_n r^n e^{in\theta}$$

Denote

$z = re^{i\theta}$

and we get $f(z) = \sum_{n\geq 0} a_n z^n$, $z\in D$

(we recover the Taylor expansion of $f$ on $D$)

Consider now $f:\overline{D}\to\mathbb{C}$ (where $\overline{D}\equiv\{z:|z|\leq 1\}$) a <u>continuous</u> function which is differentiable on $D\equiv\{z:|z|<1\}$. We defined above

$$g_r(\theta)=f(re^{i\theta}),\quad 0\leq r<1,\ \theta\in\mathbb{R}$$

Now we can define $g_1(\theta)=f(e^{i\theta})$, $\theta\in\mathbb{R}$

and we get a family $\{g_r\}_{0\leq r\leq 1}$ of functions on $\mathbb{R}$ (with complex values). As before

$$g_r(\theta)=\sum_{n\in\mathbb{Z}}a_n(r)e^{in\theta},\quad \theta\in\mathbb{R}\quad \text{(its Fourier series)},\ 0\leq r<1$$

$$g_1(\theta)\sim\sum a_n(1)e^{in\theta},\quad \theta\in\mathbb{R}\ \text{(its Fourier series)};$$

we cannot assume that there is pointwise convergence.

$f$ is continuous on the closed and bounded subset $\overline{D}\subset\mathbb{C}$, so it is uniformly continuous. It follows that

$$\forall\varepsilon>0,\exists\delta>0\quad |z_0-z_1|<\delta\Rightarrow|f(z_0)-f(z_1)|<\varepsilon$$

The Fourier coefficients have a precise formula:

$$a_n(r)=\frac{1}{2\pi}\int_0^{2\pi}g_r(\theta)e^{-in\theta}d\theta=\frac{1}{2\pi}\int_0^{2\pi}f(re^{i\theta})e^{-in\theta}d\theta$$

$n\in\mathbb{Z}$, $r\in[0,1]$

Then $$|a_n(1)-a_n(r)|=\frac{1}{2\pi}\left|\int_0^{2\pi}\left[f(e^{i\theta})-f(re^{i\theta})\right]e^{-in\theta}d\theta\right|\leq$$

$$\leq\frac{1}{2\pi}\cdot 2\pi\cdot\sup_{\theta\in\mathbb{R}}\left|f(e^{i\theta})-f(re^{i\theta})\right|$$

Now, take $1-\delta<r\leq 1$. Then $|e^{i\theta}-re^{i\theta}|<\delta$, so $|f(e^{i\theta})-f(re^{i\theta})|<\varepsilon$ $\forall\theta\in\mathbb{R}$. It follows that $|a_n(1)-a_n(r)|<\varepsilon$, $\forall n\in\mathbb{Z}$ and $a_n:[0,1]\to\mathbb{C}$ is a continuous function, $\forall n\in\mathbb{Z}$, and $\lim_{r\to 1}a_n(r)=a_n(1)$, $\forall n$ (In particular, $a_n(1)=0$, $n<0$)

Consider the periodic distribution

$$\left\langle Q_r\in\mathcal{P}',\ Q_r\sim\sum_{n=0}^{\infty}r^ne^{in\theta},\quad 0\leq r<1\right\rangle$$

(it is a distribution, because the Fourier coefficients form a sequence of rapid decrease (see next page), so in fact it is an element $Q_r\in\mathcal{P}$, $0\leq r<1$

$$\left[\ Q_r(\theta)=\sum_0^{\infty}r^ne^{in\theta}=\frac{1}{1-re^{i\theta}},\quad \theta\in\mathbb{R},\ 0\leq r<1\ \right]$$

<u>Remark</u> given $0 \leq r < 1$, the sequence $(r^n)_{n=0}^{\infty}$ is of rapid decrease

Choose $0 < p$. Then $\frac{1}{r} > 1$, $\frac{1}{r} = (1+\delta)$, $\frac{1}{r^n} = (1+\delta)^n$.

Assume
$$\frac{n}{2} > k-1 > p-1 . \text{ Then } \quad n-k+1 > \frac{n}{2}, \quad k > p$$

We get
$$\frac{1}{r^n} = (1+\delta)^n \geq \binom{n}{k}\delta^k = \frac{n(n-1)\dots(n-k+1)}{k!}\delta^k >$$
$$> \frac{1}{k!}\left(\frac{n}{2}\right)^k > \frac{1}{k!\,2^k} n^p$$

This happens for $n > 2(p-1)$

Then we can find a constant $c(p) > 0$ such that

$$r^n < c(p)\cdot n^{-p}, \quad \forall n > 0$$

and $(r^n)_{n=0}^{\infty}$ is of rapid decrease. □

$$g_r(\theta) = \sum_{n\in\mathbb{Z}} a_n(r) e^{in\theta} = \sum_{n\geq 0} a_n(r) e^{in\theta} = \sum_{n\geq 0} a_n . r^n e^{in\theta}$$

so
$$\boxed{g_r = Q_r * g_1 , \quad 0 \leq r < 1} \qquad (*)$$

Observe that then

$$g_r(\theta) = f(re^{i\theta}) = \frac{1}{2\pi}\int_0^{2\pi} Q_r(\theta - t)\, g_1(t)\, dt = \frac{1}{2\pi}\int_0^{2\pi} \frac{1}{1-re^{i(\theta-t)}} f(e^{it})\, dt =$$
$$= \frac{1}{2\pi i}\int_0^{2\pi} \frac{f(e^{it})}{e^{it} - re^{i\theta}}\, ie^{it}\, dt = \frac{1}{2\pi i}\int_{C(0;s)} \frac{f(z)}{z - z_0}\, dz$$

when $re^{i\theta} = z_0$ and $0 \leq r < s < 1$

So we recover the Cauchy integral formula.

Now consider $(*)$ above. It gives a way to define $g_r$ $(0 \leq r < 1)$ as soon as we have $g_1$, a periodic distribution (understood as a "function" on the boundary of $D$) such that $a_n = 0$ for $n < 0$, $g_1 \sim \sum_{\mathbb{Z}} a_n e_n$

We finally get the following

**Theorem**: Let $G$ be a periodic distribution (i.e., an element $G \in \mathcal{P}'$), and assume $G \sim \sum_{n \in \mathbb{Z}} a_n e_n$, $\boxed{a_n = 0 \text{ for } n < 0}$

Define

$$g_r = G * Q_r \quad , \quad 0 \leq r < 1$$

and $f(r.e^{i\theta}) = g_r(\theta)$, $0 \leq r < 1$, $\theta \in \mathbb{R}$

Then $f : D \to \mathbb{C}$ is a holomorphic function on $D$, and

$$f(z) = \sum_0^\infty a_n z^n \ , \ z \in D$$

The family $(g_r)_{r \in [0,1[}$ converges to $G$ in the sense of $\mathcal{P}'$ when $r \to 1$.

Conversely, given $f$ holomorphic in $D$, then $f(z) = \sum_{n \geq 0} a_n z^n$, $|z| < 1$. In case $(a_n)$ is a sequence of slow growth then there is a distribution $G \in \mathcal{P}'$ such that

$$g_r(\theta) = f(re^{i\theta}) = (G * Q_r)(\theta) \ , \quad 0 \leq r < 1, \ \theta \in \mathbb{R}$$

and $G \sim \sum_{n \geq 0} a_n e_n$. So $G$ is unique and $g_r \xrightarrow[r \to 1]{\mathcal{P}'} G$.

In particular, we can choose a class of distributions $\subset \mathcal{P}'$, say $G$ with Fourier series $\sum_{n \geq 0} a_n e_n$, $(a_n) \in \ell^2$ (i.e., $\sum_0^\infty |a_n|^2 < \infty$), and the holomorphic function $f(z) = \sum_{n \geq 0} a_n z^n$, $|z| < 1$ defines a sequence of functions $(g_r)_{0 \leq r < 1}$ as above which converges to $G$ in the sense of $L^2$. Conversely, given $f$ holomorphic in $D$, $f(z) = \sum_{n \geq 0} a_n z^n$, $(a_n) \in \ell^2$ we define a distribution $G$ in this class

The set of holomorphic functions on $D$ such that $f(z) = \sum_{n \geq 0} a_n z^n$ for $|z| < 1$ form the Hilbert space $H^2(D)$.

## The wave equation

Consider the following problem

Given $G, H$ in $\mathcal{P}'$, find a family $(F_t)_{t>0} \subset \mathcal{P}'$ such that

① $\exists \left.\frac{d}{dt}F_t\right|_{t=s}$ for each $s>0$

② $\left.\frac{d^2}{dt^2}F_t\right|_{t=s} = D^2F_s$, $\forall s>0$ (W)

③ $F_t \xrightarrow{\mathcal{P}'} G$, as $t\to 0$

④ $\left.\frac{d}{dt}F_t\right|_s \xrightarrow{\mathcal{P}'} H$, as $s\to 0$

Then we have the following

**Theorem** Given $H, G$ in $\mathcal{P}'$, the problem (W) has one and only one solution $(F_t)_{t>0} \subset \mathcal{P}'$.

Proof: Let $G \sim \sum_{\mathbb{Z}} b_n e_n$, $H \sim \sum_{\mathbb{Z}} c_n e_n$

As in the case of (H), let's start by proving uniqueness: assume $F_t \sim \sum_{\mathbb{Z}} a_n(t) e_n$, $t>0$. In a similar way to (H), we get that $a_n$ is a twice continuously differentiable function in $]0,+\infty[$, $n\in\mathbb{Z}$, and

$$D^2a_n(t) = -n^2a_n(t)$$
$$a_n(0) = b_n, \quad Da_n(0) = c_n, \quad n\in\mathbb{Z}, t>0$$

There is one and only one solution to this problem:

$$\left.\begin{array}{l} a_n(t) = b_n\cos nt + \frac{1}{n}c_n\sin nt, \quad n\neq 0 \\ a_0(t) = b_0 + c_0t \end{array}\right\} t>0 \qquad (S)$$

This proves uniqueness.

To prove existence consider the sequence (S) of functions on $]0,+\infty[$.

$|a_n(t)| \leq |b_n| + \frac{1}{n}|c_n|$, $n\neq 0$, so $(a_n(t))$ is the sequence of Fourier coefficients of an element $F_t\in\mathcal{P}'$.

$a_n(t) \xrightarrow[t\to 0]{} b_n$ implies (observe the uniform control above) $F_t \xrightarrow[t\to 0]{\mathcal{P}'} G$

$|Da_n(t)| \leq |nb_n| + |c_n|$, $\forall n\in\mathbb{Z}$, $\forall t>0$

$Da_n(t) \xrightarrow[t\to 0]{} c_n$. By the previous control, $\exists \left.\frac{d}{dt}F_t\right|_{t=s}$ and has Fourier coefficients $(Da_n(s))_{n\in\mathbb{Z}}$, and

$$\left.\frac{d}{dt}F_t\right|_{t=s} \xrightarrow[s\to 0]{\mathcal{P}'} H$$

Moreover $\quad |D^2 a_n(t)| \leq |n^2 b_n| + |n c_n| \, , \quad n \in \mathbb{Z} \, , \, t>0$

By the same argument

$$\left(\frac{d^2}{dt^2}\right)(F_t)\Big|_{t=s} = D^2 F_s \quad , \ \forall s>0. \qquad \square$$

The Laplace Transform

L.1 1 The structure of the test space

$\mathcal{L} := \{ u : \mathbb{R} \to \mathbb{C}$ infinitely differentiable (= smooth) such that $\forall a \in \mathbb{R}, \forall k \geq 0, \lim_{t \to +\infty} e^{at} D^k u(t) = 0 \}$

This is the class of functions which are smooth and which approaches 0 on the right very quickly.
Then define, for $u \in \mathcal{L}$,

$$\|u\|_{k,a,M} := \sup \{ e^{at} |D^k u(t)| : t \in [M, +\infty[ \}$$

for $k = 0, 1, 2, \dots, a \in \mathbb{R}, M \in \mathbb{R}$
and consider $\mathcal{L}$ endowed with this system of seminorms

$$(\mathcal{L}, \|\cdot\|_{k,a,M} : k = 0, 1, 2, \dots, a \in \mathbb{R}, M \in \mathbb{R})$$

Convergence with respect to all $\|\cdot\|_{k,a,M}$ will be denoted by $\xrightarrow{\mathcal{L}}$.
A sequence $(u_n) \subset \mathcal{L}$ is Cauchy if it is Cauchy for each $\|\|_{k,a,M}$.

L.1.2 **Theorem** : $\mathcal{L}$ is complete

Proof: Let $(u_n)_1^\infty$ be a Cauchy sequence in $\mathcal{L}$. Fix $k, a, M$ as above.
Then $(D^k u_n|_M)_{n=1}^\infty$ is a $\|\cdot\|_\infty$ - Cauchy sequence in $C[M, +\infty[$
hence it converges to some function $g_k \in C[M, +\infty[$.
It is obvious that this procedure defines a function $g_k$ on $\mathbb{R}$ (does not depend on $M$). It is enough now to prove that $Dg_k = g_{k+1}$, $k = 0, 1, 2, \dots$ Fix $k$, and take $x_0 < x \in \mathbb{R}$.

$$D^k u_n(x) - D^k u_n(x_0) = \int_{x_0}^{x} D^{k+1} u_n(t)\, dt$$

Letting $n \to \infty$,

$$g_k(x) - g_k(x_0) = \int_{x_0}^{x} g_{k+1}(t)\, dt$$

because the convergence of $(D^{k+1} u_n)_n$ to $g_{k+1}$ is uniform in $[x_0, x]$ (take $a = 0$, $M < x_0 < x$, and consider $\|\cdot\|_{k,0,M}$).
We get $Dg_k = g_{k+1}$ as stated. □

## Remark

Instead of using three indices $K, a, M$ we can repeat the same index three times. Precisely, the following Proposition has a simple proof

L.1.3

$$\boxed{\begin{array}{l}\underline{\text{Proposition}}: \text{Let } (u_n)_1^\infty \text{ be a sequence in } \mathcal{L}. \text{ Then } u_n \xrightarrow{\mathcal{L}} u \\ \text{for some } u \in \mathcal{L} \text{ if and only if } \|u - u_n\|_{KKK} \xrightarrow{n} 0 \text{ for every } K \in \mathbb{N} \cup \{0\}\end{array}}$$

## L. 2 The structure of the Laplace distributions

(L.2.1) Let $\mathcal{L}'$ be the dual space of $\mathcal{L}$, i.e., the space of all continuous linear functionals from $\mathcal{L}$ into $\mathbb{C}$.
On $\mathcal{L}'$ we shall consider the weak $*$ - topology, and we shall talk about convergence in the $\mathcal{L}'$-sense, or $\xrightarrow{\mathcal{L}'}$, i.e.,

$$F_i \in \mathcal{L}', \; F_i \xrightarrow{\mathcal{L}'} F \in \mathcal{L}' \equiv \langle u, F_i \rangle \xrightarrow[i]{} \langle u, F \rangle, \; \forall u \in \mathcal{L}$$

(L.2.2) Examples of elements in $\mathcal{L}'$

1. Fix $x_0 \in \mathbb{R}$ and consider $\delta_{x_0} : \mathcal{L} \to \mathbb{C}$. Precisely
$$\delta_{x_0}(u) = u(x_0), \; \forall u \in \mathcal{L}$$
It is obvious that $\delta_{x_0} \in \mathcal{L}'$

2. Fix $z \in \mathbb{C}$ and define
$$F_z : \mathcal{L} \to \mathbb{C}$$
$$F_z(u) = \int_0^{+\infty} e^{zt} \cdot u(t)\, dt, \; \forall u \in \mathcal{L}.$$

$F_z$ is well defined and is an element in $\mathcal{L}'$: Fix $z \in \mathbb{C}$. Choose $a \in \mathbb{R}$ such that $\operatorname{Re} z < a$. Then

$$|e^{zt} u(t)| = e^{\operatorname{Re} z \cdot t} |u(t)| = e^{at} |u(t)|\, e^{(\operatorname{Re} z - a)t} \le C \cdot e^{(\operatorname{Re} z - a)t}$$

for some constant $C > 0$.
Then $F_z(u)$ is well defined.
In order to check that it is continuous, it is useful to consider the next

(L.2.3) **Proposition**: Let $F : \mathcal{L} \to \mathbb{C}$ a linear mapping. Then $F$ is continuous if and only if there exists a constant $C > 0$ and constants $K = 0, 1, 2, \dots$, $M \in \mathbb{R}$, $a \in \mathbb{R}$ such that
$$|F(u)| \le C \cdot \|u\|_{K,a,M}, \; \forall u \in \mathcal{L}$$

Proof: One direction is clear. Assume now that the condition is not true. Then, given $K = 0, 1, 2, \dots$, we can find $V_K \in \mathcal{L}$ such that
$$|F(V_K)| \ge K \|V_K\|_{K,K,-K} \neq 0, \quad K = 1, 2, 3, \dots$$

Define
$$U_K = \frac{1}{K \|V_K\|_{K,K,-K}} \cdot V_K$$

Then
$$\|U_K\|_{K,K,-K} = \frac{1}{K}, \quad |F(U_K)| \ge 1$$
By (L.1.3) we get a contradiction. □

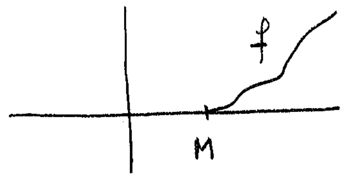
we shall check now that $F_z$ is continuous. Fix $\mathrm{Re}\, z < a$

$$\left|F_z(u)\right| = \left|\int_0^{+\infty} e^{-zt}\cdot u(t)\,dt\right| \leq \int_0^{+\infty} e^{at}\,|u(t)|\cdot e^{-(\mathrm{Re}\, z - a)t}\,dt \leq$$

$$\leq \|u\|_{0,a,0}\cdot \int_0^{+\infty} e^{-(\mathrm{Re}\, z - a)t}\,dt = k\|u\|_{0,a,0}, \quad \forall u \in \mathcal{L}.$$

(3) Fix $f : \mathbb{R} \to \mathbb{C}$ a continuous function such that for some $a \in \mathbb{R}$ and $M \in \mathbb{R}$

$$(*) \quad \boxed{\begin{array}{l} f(t) = 0, \ \forall t \leq M \\ e^{-at} f(t) \text{ is bounded} \end{array}}$$




Then define

$$F_f(u) = \int_{-\infty}^{\infty} f(t)\,u(t)\,dt \quad , \forall u \in \mathcal{L}$$

$F_f$ is well defined and belongs to $\mathcal{L}'$: Let $a < a'$. then

$$\left|F_f(u)\right| \leq \int_M^{+\infty} |f(t)\cdot e^{-at}|\cdot |e^{(a-a')t}|\cdot |u(t)\cdot e^{a't}|\,dt \leq$$

$$\leq C_1\cdot \|u\|_{0,a',M}\cdot C_2$$

for some constants $C_1 > 0$, $C_2 > 0$. Apply now (L.2.3). □

Elements in $\mathcal{L}'$ of type $F_f$ are called «functions».

In general, elements in $\mathcal{L}'$ are called <u>Laplace distributions</u>.

L3

## Laplace Transform of functions

Consider a function $f:\mathbb{R}\to\mathbb{C}$ of the class considered above: functions $f$ which define an element $F_f\in\mathcal{L}'$. Precisely

L3.1

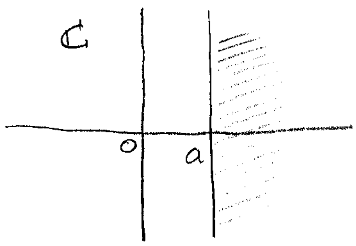
$$\left[\begin{array}{l} f:\mathbb{R}\to\mathbb{C} \text{ is continuous} \\ \exists M\in\mathbb{R} \text{ such that } f(t)=0,\ \forall t\leq M \\ \exists a\in\mathbb{R} \text{ such that } |f(t)|\leq C.e^{at},\ \forall t\in\mathbb{R} \\ \exists C\geq 0 \end{array}\right] \quad (*)$$

L.3.2 Define $e_z:\mathbb{R}\to\mathbb{C}$, $e_z(t)=e^{-zt}$, $\forall t\in\mathbb{R}$, where $z\in\mathbb{C}$ is given. Then the following function of complex variable is well defined

$$(Lf)(z)\equiv\int_{-\infty}^{\infty}e_z(t)f(t)dt=\int_{-\infty}^{\infty}e^{-zt}.f(t)\,dt$$

for $\operatorname{Re}z>a$.

It is called the <u>Laplace Transform of f</u>




L.3.3

**Theorem**: Given a function $f$ which satisfies $(*)$, $Lf$ is a holomorphic function on $\operatorname{Re}z>a$. Its derivative is

$$(Lf)'(z)=-\int_{-\infty}^{\infty}t.e^{-zt}f(t)dt$$

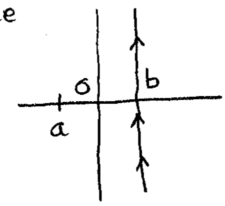
Moreover
$$|(Lf)(z)|\leq K\frac{1}{(\operatorname{Re}z-a)}e^{M(a-\operatorname{Re}z)},\ \forall \operatorname{Re}z>a$$

where $K$ is some constant.

L3.4

**Theorem** (Inversion formula): Given a function which satisfies $(*)$ let $g=Lf$. Given $b>\max\{a,0\}$, let $C$ be the line

$$\{z:\operatorname{Re}z=b\}$$




Define $$F(t)=\frac{1}{2\pi i}\int_C e^{zt}\,\frac{g(z)}{z^2}dz$$

Then $f=F''$.

## L.4. Laplace Transform of distributions

L.4.1 To extend the calculus to distributions we follow the same idea as in the case of periodic distributions : we carry the weight of the computations on the test functions

Given $F \in \mathcal{L}'$, we know that there exists $K = 0, 1, 2, \ldots$, $C > 0$, $a \in \mathbb{R}$ and $M \in \mathbb{R}$ such that

$$|F(u)| \leq C \|u\|_{K,a,M} \quad , \forall u \in \mathcal{L} \qquad [1]$$

Look at the definition of the Laplace transform of a function $f$ of class (*) (see L.3) :

$$(Lf)(z) = \int_{-\infty}^{\infty} e_z(t)\, f(t)\, dt \quad , \quad \operatorname{Re} z > a$$

when $f(t) = 0,\ \forall t \leq M,\ |f(t)| \leq C \cdot e^{at},\ \forall t$

It is tempting to define, given $F \in \mathcal{L}'$,

$$(LF)(z) = \langle e_z, F \rangle = F(e_z), \quad \operatorname{Re} z > a,$$

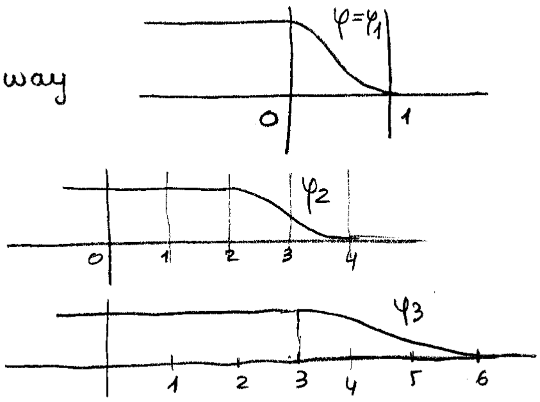
when $|F(u)| \leq C \|u\|_{K,a,M}$, $\forall u \in \mathcal{L}$.

The problem is that $e_z$ is <u>not</u> an element in $\mathcal{L}$

There is a way to circumvent this difficulty : observe first that given $F \in \mathcal{L}'$, we can fix $K$, $a$, and $M$, such that [1] holds. This reduces the problem to a certain seminorm $\|\cdot\|_{K,a,M}$. We cannot expect to find an «approximation» of $e_z$ in $\mathcal{L}$, a sequence in $\mathcal{L}$, say $(u_n)_1^\infty$, such that $u_n \xrightarrow{\mathcal{L}} e_z$, because $\mathcal{L}$ is complete and then a Cauchy sequence $(u_n)$ in $\mathcal{L}$ converges to an element in $\mathcal{L}$. However, we are using only one seminorm, and then the approximation is possible :

L.4.2

> <u>Proposition</u> : Let $a \in \mathbb{R}$ and $\operatorname{Re} z > a$. There is a sequence $(u_n)_{n=1}^{\infty}$ in $\mathcal{L}$ such that
>
> $$\|u_n - e_z\|_{K,a,M} \xrightarrow{n\to\infty} 0$$
>
> for $K = 0, 1, 2, \ldots$, and $M \in \mathbb{R}$




Proof. Let $\varphi$ be a smooth function in this way

Define $\varphi_n(t)=\varphi(t/n)$, $t\in\mathbb{R}, n=1,2,3,\dots$

$$U_n=\varphi_n\cdot e_z \quad , \quad n=1,2,3,\dots$$

Then $U_n$ is smooth, and its support is in $]-\infty, 2n]$

We shall prove that, given $K=0,1,2,\dots$ and $M\in\mathbb{R}$,

$$\|U_n-e_z\|_{K,a,M}\to 0$$

We deal with the case $K=1$. Others are treated similarly

$$e^{at}(DU_n(t)-De_z(t))=e^{at}\left[D\varphi_n(t)e_z(t)+\varphi_n(t)De_z(t)-De_z(t)\right]=$$

$$= e^{(a-z)t}z\left[1-\varphi_n(t)\right]+e^{(a-z)t}D\varphi_n(t)$$

$$\left|e^{at}(DU_n(t)-De_z(t))\right|\leq e^{(a-\mathrm{Re}\,z)t}C$$

where $C>0$ is a constant (independent of $t$ and $n$)

Observe now that $1-\varphi_n$ and $D\varphi_n$ vanish outside of $[n,2n]$.

Then

$$\left|e^{at}(DU_n(t)-De_z(t))\right|\leq C\,e^{n(a-\mathrm{Re}\,z)}$$

and this proves the assertion □

L.4.3 Now, given $F\in\mathcal{L}'$ such that $|F(u)|\leq c\|u\|_{K,a,M}$, $\forall u\in\mathcal{L}$, define

$$LF(z)=F(e_z) \quad , \quad \mathrm{Re}\,z>a$$

where $F(e_z):=\lim_{n\to\infty}F(U_n)$ , $(U_n)$ defined in (L.4.2).

and say that $LF$ is the Laplace transform of the Laplace distribution $F$

L.4.4 Some properties : given $F,G\in\mathcal{L}'$, $b\in\mathbb{C}$, consider $LF$ and $LG$ on the common domain of definition. Then

1. $L$ is linear : $L(F+G)=LF+LG$ , $L(bF)=bLF$
2. $L(\tau_s F)(z)=e^{-zs}LF(z)$
3. $L(D^kF)(z)=z^kLF(z)$
4. $L(S_-F)(z)=\frac{1}{z}LF(z)$, $z\neq 0$
5. $L(e_w\cdot F)(z)=LF(z+w)$
6. $L(F_f)=F_{Lf}$

where, as it is usual, operations on distributions take place on test functions:

translation: $T_s F(u) = F(T_{-s} u)$

conjugation: $F^*(u) = F(u^*)^*$.

$\mathrm{Re} F = \frac{1}{2}(F + F^*)$

$\mathrm{Im} F = \frac{1}{2i}(F - F^*)$

Derivation: $D^k F(u) = (-1)^k F(D^k u)$

Integration: $S_- F(u) = - F(S_+ u)$

where
$$S_+ u(t) = -\int_t^{+\infty} u(s)\,ds$$

L.4.5

**Theorem.** Given $F \in \mathcal{L}'$ such that $|F(u)| \le C \|u\|_{K,a,M}$, $\forall u \in \mathcal{L}$ then $LF$ is holomorphic in $\mathrm{Re}\, z > a$.
Moreover
$$F = D^{K+2} F_f$$
where
$$f(t) = \frac{1}{2\pi i} \int_C e^{zt} \frac{LF(z)}{z^{K+2}}\, dz$$
and $C$ is the line $\{z : \mathrm{Re}\, z = b\}$ for some $b > \max\{a, 0\}$

L.4.6

**Theorem**: Let $g$ holomorphic in $\mathrm{Re}\, z > a$. Then there exists $F \in \mathcal{L}'$ such that $LF = g$ if and only if $\exists\, C > 0$, $\exists\, K = 0, 1, 2, \dots$ $\exists\, M \in \mathbb{R}$
$$|g(z)| \le C (1 + |z|)^K \exp(-M \,\mathrm{Re}\, z), \quad \text{for } \mathrm{Re}\, z > a$$

## L.5 Solving linear differential equations

Given a polynomial

$$p(z) = \sum_{k=0}^{n} a_k z^k \qquad (a_k \in \mathbb{C},\ k = 0, 1, 2, \ldots, n)$$

define a differential operator

$$p(D) = \sum_{k=0}^{n} a_k D^k$$

L.5.1

**Theorem**: Given $H \in \mathcal{L}'$, there exists one and only one $F \in \mathcal{L}'$ such that

$$p(D) F = H \qquad [1]$$

Proof: By (L.4.5) we see that a distribution $F \in \mathcal{L}'$ is uniquely defined by its Laplace transform $LF$. Then we can apply $L$ to [1] and obtain an equivalent equation

$$L[p(D)F](z) = LH(z), \quad \forall z \text{ such that } \operatorname{Re} z > a$$

$(|H(u)| \leq C \|u\|_{K,a,M},\ \forall u \in \mathcal{L})$.

Then

$$p(z)\, LF(z) = LH(z), \quad \operatorname{Re} z > a$$

Choose $a$ big enough to get $p(z) \neq 0$, $\forall z$ such that $\operatorname{Re} z > a$.

Define

$$g(z) = \frac{LH(z)}{p(z)},$$

a holomorphic function in $\operatorname{Re} z > a$.

It satisfies the estimate in (L.4.6). Hence there exists a unique $F \in \mathcal{L}'$ such that $LF = g$ and we get [1]. $\square$