UNIVERSITAT POLITÈCNICA DE VALÈNCIA

Introduction to partial differential equations

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Outline

First order partial differential equations

- In many geometry, physics or engineering problems there are equations that involve a function, more than one indepndent variable and the partial derivatives of this function.
- A relation of this kind is called a partial differential equation.

$$F\left(x, y, u, \frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}, \frac{\partial^2 u}{\partial x^2}, \dots\right) = 0$$

- Different notations for the partial derivatives are:

$$\frac{\partial u}{\partial x}, u_x, \partial_x u, D_x u$$
$$\frac{\partial^2 u}{\partial x^2}, u_{xx}, \partial_{xx} u, D_x^2 u$$
$$\frac{\partial^2 u}{\partial x \partial y}, u_{xy}, \partial_{xy} u, D_x D_y u$$

First order partial differential equations

- First, we will focus on first order partial differential equations. Examples are given by:

$$u_t + u_x = 0$$
$$u_t + u\, u_x = 0$$
$$u_t + u\, u_x = u$$
$$3u_x - 2u_y + u = x$$

- A linear first order partial differential equation is of the form

$$\boxed{a(x,y)u_x + b(x,y)u_y + c(x,y)u = f(x,y)}$$

- The quasilinear first order partial differential equation is of the form

$$\boxed{a(x,y,u)u_x + b(x,y,u)u_y = f(x,y,u)}$$

Note that the u-term was absorbed by $f(x,y,u)$.

First order partial differential equations

Let's consider the linear first order constant coefficient partial differential equation

$$a\,u_x + b\,u_y + c\,u = f(x,y)$$

We consider the case $b = 0$

$$u_x + \frac{c}{a}\,u = \frac{f(x,y)}{a}$$

whose solution is

$$\boxed{u(x,y) = \frac{1}{a}\int_0^x f(\tau,y)e^{\frac{c}{a}(\tau - x)}\,d\tau + g(y)e^{-\frac{c}{a}x}}$$

First order partial differential equations

Now, we consider

$$a\,u_x + b\,u_y + c\,u = f$$

We consider the change of variables

$$\begin{aligned} \eta &= bx - ay \\ \xi &= y \end{aligned} \qquad \begin{aligned} x &= \frac{1}{b}\,(\eta + a\xi) \\ y &= \xi \end{aligned}$$

and the equation becomes

$$\boxed{b\,u_\xi + cu = \tilde{f}}$$

which can be solved as in the other case.

Exercise

Find the general solution of

$$3u_x - 2u_y + u = x$$

Solution

$$u(x,y) = x - 3 + c\,(-2x - 3y)\,e^{\frac{y}{2}}$$

First order partial differential equations

- We call current associated with a given velocity field $\vec{v} = (v_x, v_y)$ to a scalar field $\psi(x, y)$, such that

$$\vec{\nabla}\psi \, \vec{v} = 0$$

- The level curves associated with the surface $z = \psi(x, y)$ are tangent to the velocity field.
- To determine $\psi(x, y)$ one possibility is to impose

$$\boxed{\frac{\partial \psi}{\partial x} = -v_y \ , \quad \frac{\partial \psi}{\partial y} = v_x}$$

Example

Given the velocity field

$$v_x = x \ , \ \ v_y = -y,$$

compute the current field.

Solution

We have

$$\frac{\partial \psi}{\partial x} = y, \ \ \frac{\partial \psi}{\partial y} = x,$$

thus,

$$\psi(x, y) = xy + f(x), \ \ \psi(x, y) = xy + g(y),$$

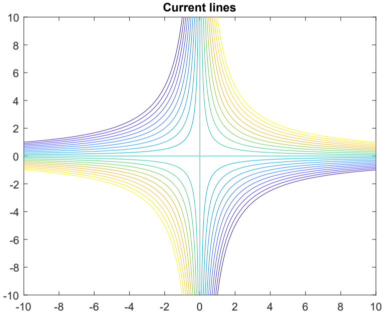
This implies $f(x) = g(y) = C$, and

$$\psi(x, y) = xy + C$$

First order partial differential equations

On possible solution is

$$\psi(x, y) = xy$$



First order partial differential equations

- Given a velocity field $\vec{v}$, we call potential function associated with this field, if it exists, to a scalar function $\phi(x, y)$, such that

$$\boxed{\vec{\nabla}\phi = \vec{v}}$$

- A condition for the velocity field to have a potential function is

$$\frac{\partial v_y}{\partial x} = \frac{\partial v_x}{\partial y}$$

Example

Given the velocity field

$$v_x = x \ , \quad v_y = -y,$$

compute the potential of velocities.

Solution
We have

$$\frac{\partial \phi}{\partial x} = x \ , \quad \frac{\partial \phi}{\partial y} = -y \ ,$$

that is

$$\phi(x, y) = \frac{1}{2}x^2 + f(y), \quad \phi(x, y) = -\frac{1}{2}y^2 + g(x),$$

Computing the derivatives

$$\frac{\partial \phi}{\partial y} = -y = f'(y) \ , \quad \frac{\partial \phi}{\partial x} = x = g'(x) \ ,$$

First order partial differential equations

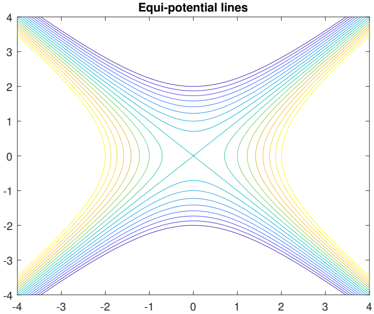
This implies

$$f(y) = -\frac{1}{2}y^2 + C_1, \quad g(x) = \frac{1}{2}x^2 + C_2,$$

and

$$\phi(x, y) = \frac{1}{2}\left(x^2 - y^2\right) + C$$

The equipotential lines



First order partial differential equations

Exercise

Given the velocity field

$$\vec{v}(x, y) = \Big(y\cos(x) + y^2, \sin(x) + 2xy - 2y\Big)$$

Find its potential function.

Solution

$$\phi(x, y) = y\sin(x) + y^2x - y^2 + k$$

The method of characteristics

We start from a quasilinear partial differential equation

$$\boxed{a(x,y,u)u_x + b(x,y,u)u_y = c(x,y,u)}$$

If $u = u(x,y)$ is a solution of the equation,

$$f(x,y,u) = u(x,y) - u = 0$$

describes the solution surface. The normal vector associated with this surface is

$$\vec{\nabla} f = (u_x, u_y, -1)$$

Introducing the vector $\vec{v} = (a,b,c)$, the differential equation can be written as

$$\boxed{\vec{v}\,\vec{\nabla} f = 0}$$

The method of characteristics

- This implies that the vector $\vec{v} = (a, b, c)$ is tangent to the solution surface. Geometrically, $\vec{v}$ defines a direction field, called the characteristic field.
- Recall that one can parametrize space curves,

$$\vec{r}(t) = (x(t), y(t), u(t)), \quad t \in [t_1, t_2] .$$

- The tangent to this curve is

$$\vec{v} = \frac{d\vec{r}}{dt} = \left(\frac{dx}{dt}, \frac{dy}{dt}, \frac{du}{dt} \right)$$

These are called the characteristic curves,

$$\frac{dx}{dt} = a, \quad \frac{dy}{dt} = b, \quad \frac{du}{dt} = c$$

These relations can be summarized as

$$\boxed{\frac{dx}{a} = \frac{dy}{b} = \frac{du}{c}}$$

The method of characteristics

Theorem

The general solution of a quasilinear partial differential equation

$$a(x,y,u)u_x + b(x,y,u)u_y = c(x,y,u)$$

is of the form $F(f,g) = 0$, where $f(x,y,u) = c_1$ and $g(x,y,u) = c_2$ are two independent solutions of the characteristic system

$$\frac{dx}{a} = \frac{dy}{b} = \frac{du}{c}$$

The method of characteristics

Example

Obtain the general solution of the equation

$$x\frac{\partial u}{\partial x} + u\frac{\partial u}{\partial y} = y$$

Solution

The characteristic system is

$$\frac{dx}{x} = \frac{dy}{u} = \frac{du}{y}$$

Considering the equation

$$\frac{dy}{u} = \frac{du}{y} \quad \rightarrow \quad y^2 - u^2 = c_1$$

And if

$$\frac{dy}{u} = \frac{du}{y} = \frac{dx}{x} \quad \rightarrow \quad \frac{dy+du}{u+y} = \frac{dx}{x}$$

We have

$$\ln(x) = \ln(u+y) + \ln(c_2)$$

that is

$$c_2 = \frac{x}{u+y}$$

and the general solution of the equation is

$$F(c_1, c_2) = 0 \quad \rightarrow \quad F\left(y^2 - u^2, \frac{x}{u+y}\right) = 0$$

Example

Find the solution of the equation

$$3\frac{\partial u}{\partial x} - 2\frac{\partial u}{\partial y} = x - u$$

that satisfies $u(x, x) = x$.

Solution

The characteristic system is

$$\frac{dx}{3} = \frac{dy}{-2} = \frac{du}{x - u}$$

Considering the equation

$$\frac{dx}{3} = \frac{dy}{-2} \quad \rightarrow \quad 2x + 3y = c_1$$

The method of characteristics

And if

$$\frac{dx}{3} = \frac{du}{x-u} \quad \rightarrow \quad \frac{du}{dx} = \frac{1}{3}(x-u)$$

whos solution is

$$u = x - 3 + c_2 e^{-\frac{x}{3}}$$

Since the solution is $F(c_1, c_2) = 0$ this implies $c_2 = G(c_1) = G(2x + 3y)$ and

$$u(x,y) = x - 3 + G(2x+3y)\, e^{-\frac{x}{3}}$$

Since $u(x,x) = x$,

$$x = x - 3 + G(5x) e^{-\frac{x}{3}}$$

and

$$G(5x) = 3e^{\frac{x}{3}} \quad \rightarrow \quad G(x) = 3e^{\frac{x}{15}}$$

The solution is

$$\boxed{u(x,y) = x - 3 + 3e^{\frac{y-x}{5}}}$$

The method of characteristics

Example

Solve the advection equation

$$\frac{\partial u}{\partial t} + c\frac{\partial u}{\partial x} = 0$$

being c a constant.

Solution

The characteristic system

$$\frac{dt}{1} = \frac{dx}{c} = \frac{du}{0} = d\tau$$

this means that the characteristic curves satisfy

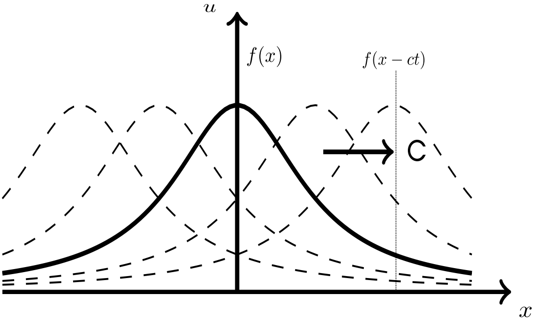
$$\frac{dx}{d\tau} = c, \quad \frac{du}{d\tau} = 0$$

The method of characteristics

this is $u = c_1$ and $x = ct + c_2$ and the general solution

$$F(u, x - ct) = 0 \quad \rightarrow \quad u = f(x - ct)$$

where F is an arbitrary function.
This solution is a travelling wave



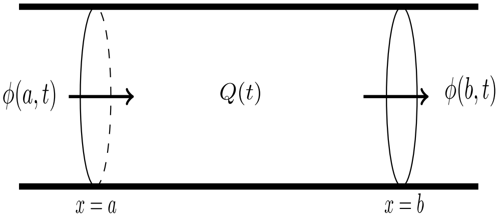
It is satisfied that

$$\begin{aligned} 0 &= u_t + cu_x \\ &= \frac{\partial u}{\partial t} + \frac{dx}{dt}\frac{\partial u}{\partial x} = \frac{du(x(t),t)}{dt} \end{aligned}$$

This implies that $u(x,t)$ is constant along the characteristics.

Conservation laws

- Conservation laws are useful in modelling several systems.
- They determine the rate of change of some quantity, $Q(t)$, in a region, $a \leq x \leq b$.
- They describe the fluid flowing in one dimension, such as water flowing in a stream. Or, it could be the transport of mass, such as a pollutant. One could think of traffic flow down a straight road.



Conservation laws

- The rate of change of $Q(t)$ is given as
  the rate of change of Q = *Rate in - Rate Out + source term.*
- We can describe this flow in terms of the flux, $\phi(x,t)$ over the ends of the region. On the left side we have a gain $\phi(a,t)$ and on the right side of the region there is a loss of $\phi(b,t)$.

$$\frac{dQ}{dt} = \phi(a,t) - \phi(b,t) + \int_a^b f(x,t)\,dx$$

where $f(x,t)$ is the source density. This can be rewritten as

$$\frac{dQ}{dt} = -\int_a^b \frac{\partial \phi(x,t)}{\partial x}\,dx + \int_a^b f(x,t)\,dx$$

Conservation laws

- Introducing the density function $u(x,t)$

$$Q(t) = \int_a^b u(x,t)\,dx$$

we have

$$\frac{d}{dt}\int_a^b u(x,t)\,dx = -\int_a^b \frac{\partial \phi(x,t)}{\partial x}\,dx + \int_a^b f(x,t)\,dx$$

This is

$$\int_a^b \left(u_t(x,t) + \phi_x(x,t) - f(x,t)\right)\,dx = 0$$

- One possibility is

$$\boxed{u_t(x,t) + \phi_x(x,t) - f(x,t) = 0}$$

which is a local conservation law.

- We can write

$$\frac{\partial \phi}{\partial x} = \frac{d\phi}{du}\frac{\partial u}{\partial x}$$

and

$$u_t(x,t) + \phi'(u)u_x(x,t) - f(x,t) = 0$$

Example

The inviscid Burgers' equation is given when $\phi = \frac{1}{2}u^2$ and $f(x,t) = 0$

$$u_t + uu_x = 0$$

which is also called a nonlinear advection equation.

Traffic flow

Let $u(x,t)$ be the density of cars. Let $\phi(x,t)$ denote the number of cars per hour passing position x at time t. The flux in this model is $\phi = uv$, where v is the velocity of the cars at position x and time t.
We need to assume a relationship between the car velocity and the car density. Let's assume the simplest form, a linear relationship,

$$v = v_1 - \frac{v_1}{u_1}u$$

We can now write the equation for the car density,

$$\begin{aligned} 0 &= u_t + \phi' u_x \\ &= u_t + v_1\left(1 - \frac{2u}{u_1}\right)u_x \end{aligned}$$

Nonlinear advection equation

- Given the linear advection equation

$$u_t + cu_x = 0$$

its characteristic lines satisfy

$$\frac{dx}{dt} = c, \quad \frac{du}{dt} = 0$$

then $u(x, t)$ is constant along the characteristics $x = x_0 + ct$.

- The nonlinear advection equation is

$$u_t + c(u)u_x = 0$$

The characteristic lines satisfy

$$\frac{dx}{dt} = c(u), \quad \frac{du}{dt} = 0$$

this is $u(x, t)$ is constant along the characteristics $x'(t) = c(u)$.

Nonlinear advection equation

Example

Solve

$$u_t + uu_x = 0$$

with the initial condition $u\left(x,0\right) = e^{-x^2}$.

Solution

The characteristic system

$$\frac{du}{dt} = 0, \quad \frac{dx}{dt} = u$$

Since u is a constant, on the characteristics,

$$x = x_0 + u\left(x_0\right)t = x_0 + te^{-x_0^2}$$

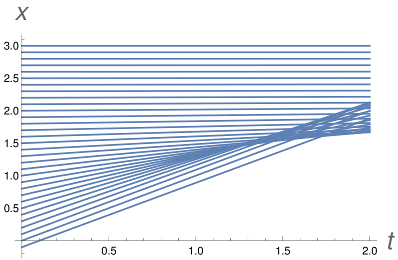
The solution

$$\boxed{u = e^{-x^2} \quad \text{along} \quad x = x_0 + te^{-x_0^2}}$$

Nonlinear advection equation

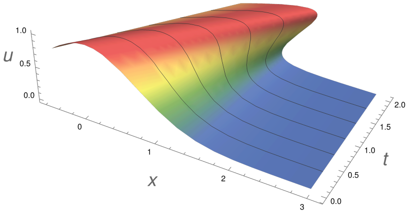
We consider the initial condition $u_0(x) = e^{-(2(x-1))^2}$

```
Clear[f0, fval, x, f]
f0[x_] = Exp[-(2 (x - 1))^2];
(* Condición inicial *)
x[t_,x0_] = x0 + f0[x0] t;
f[t_,x0_] = f0[x0];
fval[t_] := Table[{x[t, x0], f[t, x0]}, {x0, -.5, 3, .1}]
Plot[Table[x[t, x0], {x0, -.5, 3, .1}], {t, 0, 2},
AxesLabel -> {Text[Style["t", Italic, 23]],
Text[Style["x", Italic, 23]]}]
```

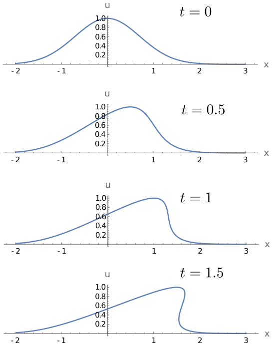


Nonlinear advection equation

```
ParametricPlot3D[{x[t, x0], t, f[t, x0]}, {x0, -.5, 3}, {t, 0, 2},
ColorFunction -> "DarkRainbow",
PlotStyle -> Directive[Opacity[0.9]], MeshFunctions -> {#2 &},
Mesh -> 5, PlotRange -> All, Axes -> {True, True, True} ,
Boxed -> False, ImageSize -> 600,
AxesLabel -> {Text[Style["x", Italic, 23]],
        Text[Style["t", Italic, 23]], Text[Style["u", Italic, 23]]}]
```



Nonlinear advection equation



The initial profile propagates to the right with the higher points traveling faster than the lower points. Around $t = 1.0$ the wave breaks and becomes multivalued. The time at which the function becomes multivalued is called the breaking time.

Breaking time

Given the nonlinear diffusion equation

$$\partial_t u + c(u)\partial_x u = 0$$

we have that the wave speed is

$$F\left(x_0\right) = c\left(u_0\left(x_0\right)\right)$$

and the characteristic

$$x = x_0 + tF\left(x_0\right)$$

The solution

$$u(x,t) = u\left(0, x_0\right) = u_0\left(x_0\right) \quad \text{along} \quad x = x_0 + tF\left(x_0\right)$$

Breaking time

We have

$$\frac{\partial u}{\partial x} = \frac{\partial u}{\partial x_0}\frac{\partial x_0}{\partial x}, \quad \frac{\partial u}{\partial t} = \frac{\partial u}{\partial x_0}\frac{\partial x_0}{\partial t}$$

From

$$x_0 = x - tF\left(x_0\right)$$

we obtain

$$\begin{aligned} \frac{\partial x_0}{\partial x} &= 1 - tF'\left(x_0\right)\frac{\partial x_0}{\partial x} \\ &= \frac{1}{1 + tF'\left(x_0\right)} \end{aligned}$$

$$\begin{aligned} \frac{\partial x_0}{\partial t} &= -F\left(x_0\right) - tF'\left(x_0\right)\frac{\partial x_0}{\partial t} \\ &= -\frac{F\left(x_0\right)}{1 + tF'\left(x_0\right)} \end{aligned}$$

Breaking time

$\frac{\partial x_0}{\partial x}$ and $\frac{\partial x_0}{\partial t}$ are undefined if

$$1 + tF'(x_0) = 0$$

that is

$$t = -\frac{1}{F'(x_0)}$$

The breaking time

$$t_b = \text{mín}\left\{-\frac{1}{F'(x_0)}\right\}$$

Breaking time

Example

Find the breaking time for

$$u_t + uu_x = 0$$

with $u(x,0) = e^{-x^2}$.

Solution

We have that

$$\begin{aligned} F(x_0) &= e^{-x_0^2} \\ F'(x_0) &= -2x_0 e^{-x_0^2} \end{aligned}$$

thus,

$$t = \frac{1}{2x_0 e^{-x_0^2}}$$

Breaking time

To find the minimum

$$\frac{dt}{dx_0} = \frac{d}{dx_0}\left(\frac{e^{x_0^2}}{2x_0}\right) = \left(2 - \frac{1}{x_0^2}\right)\frac{e^{x_0^2}}{2} = 0$$

this implies $x_0 = \frac{1}{\sqrt{2}}$ and

$$t_b = t\left(\frac{1}{\sqrt{2}}\right) = \frac{1}{\frac{2}{\sqrt{2}}e^{-\frac{1}{2}}} = \sqrt{\frac{e}{2}} \approx 1.16$$

Second order differential equations

Examples of second order differential equations:

Wave equation

$$\frac{1}{c^2}\frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial x^2}.$$

Laplace's equation

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0.$$

Diffusion equation

$$\frac{1}{c^2}\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}.$$

Second order differential equations

- A second-order PDE is called linear if it is linear in the derivatives and in the function itself, otherwise it is called nonlinear
- The general form of a linear partial differential equation with two independent variables x and y is the following one

$$A\frac{\partial^2 z}{\partial x^2} + B\frac{\partial^2 z}{\partial x \partial y} + C\frac{\partial^2 z}{\partial y^2} + D\frac{\partial z}{\partial x} + E\frac{\partial z}{\partial y} + Fz = G,$$

  with $A = A(x, y)$, $B = B(x, y)$, $C = C(x, y)$, $D = D(x, y)$, $E = E(x, y)$, $F = F(x, y)$ and $G = G(x, y)$.
- If $G(x, y) = 0$ it is called homogeneous otherwise non-homogeneous.

Second order differential equations

- A linear second-order PDE of constant coefficients $\Phi(D_x, D_y)z = 0$ is reducible, if it can be written in the form

$$\varphi_1(D_x, D_y)\left[\varphi_2(D_x, D_y)\right] z = \varphi_2(D_x, D_y)\left[\varphi_1(D_x, D_y)\right] z = 0,$$

being

$$\varphi_i(D_x, D_y) = a_i D_x + b_i D_y + c_i, \ \ i = 1, 2.$$

- To solve this kind of equations, we have to take into account that it has to be satisfied

$$\left(a_i D_x + b_i D_y + c_i\right) z = 0, \quad i = 1, 2,$$

- We consider the characteristic system

$$\frac{dx}{a_i} = \frac{dy}{b_i} = -\frac{dz}{c_i z}, \quad i = 1, 2.$$

Second order differential equations

- For $i = 1, 2$ we have

$$\begin{cases} \dfrac{dx}{a_i} = \dfrac{dy}{b_i} \\ \\ \dfrac{dx}{a_i} = -\dfrac{dz}{c_i z} \end{cases} \rightarrow \begin{cases} a_i y - b_i x = C_1 \\ \\ z e^{\dfrac{c_i x}{a_i}} = C_2 \end{cases} .$$

- The general solution

$$z = e^{-\dfrac{c_i x}{a_i}} \psi_i \left(a_i y - b_i x \right),$$

With $\psi_i\left(.\right)$, $i = 1, 2$ arbitrary functions.
That is, the general solution is

$$\boxed{z = e^{-\dfrac{c_1 x}{a_1}} \psi_1 \left(a_1 y - b_1 x \right) + e^{-\dfrac{c_2 x}{a_2}} \psi_2 \left(a_2 y - b_2 x \right).}$$

Example

Find the general solution of the wave equation

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2},$$

Solution

Using the derivative operators, we can write

$$\left(D_t^2 - c^2 D_x^2\right) u = \left(D_t - cD_x\right)\left(D_t + cD_x\right) u$$

thus, it is a reducible equation. Using the charcteristic systems, we find the general solution

$$\boxed{u = \psi_1\left(x + ct\right) + \psi_2\left(x - ct\right),}$$

being $\psi_i\left(.\right)$, $i = 1, 2$ arbitrary functions. This solution is known as the D'Alembert solution.

Classification of partial differential equations

Linear second order partial differential equations (PDE) in general, or the governing equations in fluid dynamics in particular, are classified into three categories:

1. Elliptic,
2. parabolic,
3. Hyperbolic.

Let us consider the partial differential equation in a two-dimensional domain of the form

$$A\frac{\partial^2 u}{\partial x^2} + B\frac{\partial^2 u}{\partial x \partial y} + C\frac{\partial^2 u}{\partial y^2} + D\frac{\partial u}{\partial x} + E\frac{\partial u}{\partial y} + Fu + G = 0 \ .$$

Mathematically, the classification of second-order PDEs is based upon the possibility of reducing this equation by coordinate transformation to a canonical or standard form at a point.

Classification of partial differential equations

We rewrite the second order equation as

$$Au_{xx} + Bu_{xy} + Cu_{yy} = \Phi\left(x, y, u, u_x, u_y\right) \ .$$

The type of second-order PDE at a point (x_0, y_0) depends on the sign of the discriminant defined as

$$\boxed{\Delta\left(x_0, y_0\right) = \left|\begin{array}{cc} B & 2A \\ 2C & B \end{array}\right| = B^2 - 4AC \ .}$$

Classification of partial differential equations

The terminology hyperbolic, parabolic, and elliptic chosen to classify PDEs reflects the analogy between the form of the discriminant, $B^2 - 4AC$, for PDEs and the form of the discriminant, $B^2 - 4AC$, which classifies conic sections given by

$$Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0 \ .$$

The type of the curve represented by the above conic section depends on the sign of the discriminant, $\Delta \equiv B^2 - 4AC$.

- If $\Delta > 0$, the curve is a hyperbola,
- If $\Delta = 0$ the curve is an parabola,
- If $\Delta < 0$ the equation is a ellipse.

Classification of partial differential equations

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0,$$

$A = 1$, $B = 0$, $C = 1$ and $\Delta = -4 < 0$. Elliptic equation.

$$\frac{\partial u}{\partial t} - \alpha \frac{\partial^2 u}{\partial x^2} = 0, \quad (\alpha > 0),$$

$A = -\alpha$, $B = 0$, $C = 0$ and $\Delta = 0$. Parabolic equation.
For the wave equation

$$\frac{\partial^2 u}{\partial t^2} - a^2 \frac{\partial^2 u}{\partial x^2} = 0,$$

$A = 1$, $B = 0$, $C = -a^2$ and $\Delta = 4a^2 > 0$. Hyperbolic equation.

Canonical forms

The Tricomi equation

$$y\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0, \quad \Delta = -4y$$

When $y > 0$ it is elliptic, when $y = 0$ it is parabolic and when $y < 0$ it is hyperbolic.
To reduce a second order partial differential equation to a canonical form, we transform the independent variables x and y to the new independent variables ξ and η through the change of variables

$$\xi = \xi(x, y) \ , \quad \eta = \eta(x, y) \ ,$$

The Jacobian

$$J = \frac{\partial(\xi, \eta)}{\partial(x, y)} = \begin{vmatrix} \xi_x & \xi_y \\ \eta_x & \eta_y \end{vmatrix} \neq 0 \ .$$

Canonical forms

We define $w(\xi,\eta) = u(x(\xi,\eta), y(\xi,\eta))$ such that $u(x,y) = w(\xi(x,y), \eta(x,y))$. Thus,

$$
\begin{aligned}
u_x &= w_\xi \xi_x + w_\eta \eta_x \ , \\
u_y &= w_\xi \xi_y + w_\eta \eta_y \ , \\
u_{xx} &= w_{\xi\xi}\xi_x^2 + 2w_{\xi\eta}\xi_x\eta_x + w_{\eta\eta}\eta_x^2 + w_\xi \xi_{xx} + w_\eta \eta_{xx} \ , \\
u_{yy} &= w_{\xi\xi}\xi_y^2 + 2w_{\xi\eta}\xi_x\eta_x + w_{\eta\eta}\eta_y^2 + w_\xi \xi_{yy} + w_\eta \eta_{yy} \ , \\
u_{xy} &= w_{\xi\xi}\xi_x\xi_y + w_{\xi\eta}\left(\xi_x\eta_y + \xi_y\eta_x\right) + w_{\eta\eta}\eta_x\eta_y + w_\xi \xi_{xy} + w_\eta \eta_{xy} \ .
\end{aligned}
$$

Substituting in the equation

$$
a w_{\xi\xi} + b w_{\xi\eta} + c w_{\eta\eta} = \Phi\left(x, y, w, w_\xi, w_\eta\right) \ .
$$

where

$$
\begin{aligned}
a &= A\xi_x^2 + B\xi_x\xi_y + C\xi_y^2 \ , \\
b &= 2A\xi_x\eta_x + B\left(\xi_x\eta_y + \xi_y\eta_x\right) + 2C\xi_y\eta_y \ , \\
c &= A\eta_x^2 + B\eta_x\eta_y + C\eta_y^2 \ .
\end{aligned}
$$

Canonical forms

We observe that

$$\begin{pmatrix} a & b/2 \\ b/2 & c \end{pmatrix} = \begin{pmatrix} \xi_x & \xi_y \\ \eta_x & \eta_y \end{pmatrix} \begin{pmatrix} A & B/2 \\ B/2 & C \end{pmatrix} \begin{pmatrix} \xi_x & \xi_y \\ \eta_x & \eta_y \end{pmatrix}^T .$$

Thus,

$$b^2 - 4ac = J^2 \left(B^2 - 4AC \right) ,$$

this shows that the discriminant of the transformed equation has the same sign as the discriminant of the original equation.

Canonical forms

A PDE is hyperbolic if the discriminant $\Delta = B^2 - 4AC > 0$. Thus, for a hyperbolic PDE, we should have $b^2 - 4ac > 0$.
The simplest case of satisfying this condition is $a = c = 0$. We get the following canonical form of hyperbolic equation,

$$\boxed{w_{\xi\eta} = \psi\left(\xi, \eta, w, w_{\xi}, w_{\eta}\right)\ .}$$

This form is called the first canonical form of the hyperbolic equation.
We also have another simple case when $b = 0$ and $c = -a$. In this case we obtain

$$\boxed{w_{\xi\xi} - w_{\eta\eta} = \psi\left(\xi, \eta, w, w_{\xi}, w_{\eta}\right)\ ,}$$

which is the second canonical form of the hyperbolic equation.

Canonical forms

A PDE is parabolic if the discriminant $\Delta = B^2 - 4AC = 0$. that is $b^2 - 4ac = 0$. The simplest case of satisfying this condition is c (or a) $= 0$. In this case another necessary requirement $b = 0$ will follow automatically, $b^2 - 4ac = 0$.

So, we get the following canonical form of parabolic equation:

$$\boxed{w_{\xi\xi} = \psi\left(\xi, \eta, w, w_\xi, w_\eta\right)\ .}$$

Canonical forms

A PDE is elliptic if the discriminant $\Delta = B^2 - 4AC < 0$, that is $b^2 - 4ac < 0$. The simplest case of satisfying this condition is $b = 0$ and $c = a$. So, we get the following canonical form of elliptic equation:

$$\boxed{w_{\xi\xi} + w_{\eta\eta} = \psi\left(\xi, \eta, w, w_\xi, w_\eta\right)\ .}$$

Hyperbolic equations

In this case, we have seen that, to reduce this PDE to canonical form we need to choose the new variables ξ and η such that the coefficients a and c vanish

$$\begin{aligned} a &= A\xi_x^2 + B\xi_x\xi_y + C\xi_y^2 = 0 \ , \\ c &= A\eta_x^2 + B\eta_x\eta_y + C\eta_y^2 = 0 \ , \end{aligned}$$

that is,

$$A\left(\frac{\xi_x}{\xi_y}\right)^2 + B\left(\frac{\xi_x}{\xi_y}\right) + C = 0 \ ,$$

$$A\left(\frac{\eta_x}{\eta_y}\right)^2 + B\left(\frac{\eta_x}{\eta_y}\right) + C = 0 \ .$$

Hyperbolic equations

The two distinct roots of these equations are

$$\mu_1 = \frac{\xi_x}{\xi_y} = \frac{-B + \sqrt{B^2 - 4AC}}{2A} \ , \quad \mu_2 = \frac{\eta_x}{\eta_y} = \frac{-B - \sqrt{B^2 - 4AC}}{2A} \ .$$

We consider curves of the form $\xi(x, y) = k$. This implies

$$d\xi = \xi_x dx + \xi_y dy = 0 \ ,$$

that is,

$$\frac{dy}{dx} = -\frac{\xi_x}{\xi_y} \ , \quad \frac{dy}{dx} = -\frac{\eta_x}{\eta_y} \ .$$

This implies that

$$A\left(\frac{dy}{dx}\right)^2 - B\left(\frac{dy}{dx}\right) + C = 0 \ .$$

The solutions are

$$\frac{dy}{dx} = \frac{B + \sqrt{B^2 - 4AC}}{2A} = \lambda_1 \ ,$$
$$\frac{dy}{dx} = \frac{B - \sqrt{B^2 - 4AC}}{2A} = \lambda_2 \ ,$$

and the change of variables is given by

$$\boxed{\xi = y - \lambda_1 x \ , \quad \eta = y - \lambda_2 x \ .}$$

It is easy to show that the hyperbolic PDE has a second canonical form. The following linear change of variables

$$\alpha = \xi + \eta \ , \quad \beta = \xi - \eta \ ,$$

leads to the second canonical form of the hyperbolic equation.

Hyperbolic equations

For example, we consider the wave equation

$$\frac{\partial^2 u}{\partial t^2} - c^2 \frac{\partial^2 u}{\partial x^2} = 0 \ ,$$

then

$$A = 1 \ , \quad B = 0 \ , \quad C = -c^2 \ ,$$

and the discriminant $\Delta = 4c^2 > 0$ and the equation is hyperbolic.
For this case

$$\lambda_1 = c \ , \quad \lambda_2 = -c \ ,$$

and the transformation

$$\xi = x - ct \ , \quad \eta = x + ct \ ,$$

Transforms the wave equation into its canonical form

$$w_{\xi\eta} = 0 \ .$$

The general solution is

$$w(\xi, \eta) = f(\xi) + g(\eta) \ ,$$

that is,

$$u(x, t) = f(x - ct) + g(x + ct) \ .$$

Hyperbolic equations

Exercise

The so-called small disturbance potential equation:

$$\left(1 - M_\infty^2\right) \frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} = 0$$

was the form of equation used by Murman and Cole to obtain the first numerical solution for a transonic flow around an airfoil with shocks.

Show that, depending on the Mach number, the small disturbance potential equation is elliptic, parabolic, or hyperbolic. Find the characteristic variables for the hyperbolic case and hence write the equation in canonical form.

Elliptic equations

Potential equation

Another interesting example is provided by the stationary potential flow equation in two dimensions,

$$\left(1-\frac{u^2}{a^2}\right)\frac{\partial^2\phi}{\partial x^2}-\frac{2uv}{a^2}\frac{\partial^2\phi}{\partial x\partial y}+\left(1-\frac{v^2}{a^2}\right)\frac{\partial^2\phi}{\partial y^2}=0$$

where a is the speed of sound.

The discriminant

$$B^2-4AC=4\left(\frac{u^2+v^2}{a^2}-1\right)=4(M^2-1)$$

where the Mach number is $M=\sqrt{u^2+v^2}/a$.

Elliptic equations

- The stationary potential equation is elliptic for subsonic flows and hyperbolic for supersonic flows.
- Along the sonic line $M = 1$, the equation is parabolic.
- This mixed nature of the potential equation has been a great challenge for the numerical computation of transonic flows since the transition line between the subsonic and the supersonic regions is part of the solution.

Parabolic equations

For a parabolic PDE the discriminant $\Delta = B^2 - 4AC = 0$. In this case, we have seen that, to reduce this PDE to canonical form we need to choose the new variables ξ and η such that the coefficients a and b vanish. For a,

$$a = A\xi_x^2 + B\xi_x\xi_y + C\xi_y^2 = 0 \ ,$$

that is,

$$a = A\left(\frac{\xi_x}{\xi_y}\right)^2 + B\left(\frac{\xi_x}{\xi_y}\right) + C = 0 \ .$$

If we consider the coordinate line $\xi(x, y) = k$,

$$d\xi = \xi_x dx + \xi_y dy = 0 \ ,$$

and the solution is

$$A\left(\frac{dy}{dx}\right)^2 - B\left(\frac{dy}{dx}\right) + C = 0 \ .$$

This is called the characteristic polynomial of the PDE. Since $B^2 - 4AC = 0$ in this case,

$$\frac{dy}{dx} = \frac{B}{2A} = \lambda \ .$$

There is only one family of real characteristic curves. The required variables ξ is

$$\xi = y - \lambda x \ .$$

Parabolic equations

To determine the second transformation variable η, we set $b = 0$

$$\begin{aligned} b &= 2A\xi_x\eta_x + B\left(\xi_x\eta_y + \xi_y\eta_x\right) + 2C\xi_y\eta_y = 0 \ , \\ & 2A\left(-\frac{B}{2A}\right) + B\left(\left(-\frac{B}{2A}\right)\eta_y + \eta_x\right) + 2C\eta_y = 0 \ , \\ & -B\eta_x - \frac{B^2}{2A}\eta_y + B\eta_x + 2C\eta_y = 0 \ , \\ & \left(B^2 - 4AC\right)\eta_y = 0 \ . \end{aligned}$$

This means that η can be chosen arbitrarily.
The canonical form in this case is

$$w_{\xi\xi} = \psi(\xi, \eta, w, w_\xi, w_\eta) \ .$$

Parabolic equations

Example

Obtain the canonical form of the equation

$$\frac{\partial^2 u}{\partial x^2} + 4\frac{\partial^2 u}{\partial x \partial y} + 4\frac{\partial^2 u}{\partial y^2} + u = 0$$

Solution:

The discriminant $\Delta = B^2 - 4AC = 16 - 16 = 0$. Hence, the equation is parabolic.

Solving the characteristic equation

$$A\left(\frac{dy}{dx}\right)^2 - B\left(\frac{dy}{dx}\right) + C = 0\ , \rightarrow\ \lambda = 2$$

Parabolic equations

The change of variables:

$$\begin{cases} \xi = y - 2x \\ \eta = x \end{cases}$$

The canonical form is

$$\frac{\partial^2 w}{\partial \eta^2} + w = 0$$

Elliptic equations

For an elliptic PDE the discriminant $\Delta = B^2 - 4AC < 0$. In this case, we have seen that, to reduce this PDE to canonical form we need to choose the new variables ξ and η to produce $b = 0$ and $a = c$, or $b = 0$ and $a - c = 0$. Then,

$$\begin{aligned} A\left(\xi_x^2 - \eta_x^2\right) + B\left(\xi_x\xi_y - \eta_x\eta_y\right) + C\left(\xi_y^2 - \eta_y^2\right) &= 0 \\ 2A\xi_x\eta_x + B\left(\xi_x\eta_y + \xi_y\eta_x\right) + 2C\xi_y\eta_y &= 0\ . \end{aligned}$$

We add the first of these equation to complex number i times the second to give

$$A\left(\xi_x + i\eta_x\right)^2 + B\left(\xi_x + i\eta_x\right)\left(\xi_y + i\eta_y\right) + C\left(\xi_y + i\eta_y\right)^2 = 0$$

that is,

$$A\left(\frac{\xi_x + i\eta_x}{\xi_y + i\eta_y}\right)^2 + B\left(\frac{\xi_x + i\eta_x}{\xi_y + i\eta_y}\right) + C = 0$$

Elliptic equations

We consider

$$\begin{aligned}\alpha(x,y) &= \xi(x,y) + i\eta(x,y) \ ,\\ \beta(x,y) &= \xi(x,y) - i\eta(x,y) \ ,\end{aligned}$$

and the curves $\alpha(x,y) = k$, $\beta(x,y) = \tilde{k}$, which implies

$$\frac{dy}{dx} = -\frac{\alpha_x}{\alpha_y} \ , \quad \frac{dy}{dx} = -\frac{\beta_x}{\beta_y} \ ,$$

and the characteristic curves of the PDE are

$$\frac{dy}{dx} = \lambda_1 = \frac{B + i\sqrt{4AC - B^2}}{2A}, \quad \frac{dy}{dx} = \lambda_2 = \frac{B - i\sqrt{4AC - B^2}}{2A},$$

Clearly, the solution of this differential equations are necessarily complex-valued and as a consequence no real characteristic exist for an elliptic PDE.

Elliptic equations

The solutions are

$$\frac{B \pm i\sqrt{4AC - B^2}}{2A} = \{\alpha, \beta\} \ .$$

Now the real and imaginary parts of α and β give the required transformation variables ξ and η.

Thus, we have

$$\xi = y - \frac{\alpha + \beta}{2}x, \quad \eta = \frac{\alpha - \beta}{2i}x,$$

With this choice of coordinate variables the equation reduces to following canonical form

$$w_{\xi\xi} + w_{\eta\eta} = \psi(\xi, \eta, w, w_\xi, w_\eta)$$

Elliptic equations

Example

Given the equation

$$\frac{\partial^2 u}{\partial x^2} + x^2 \frac{\partial^2 u}{\partial y^2} = 0, \quad x \neq 0$$

Obtain its canonical form.

Solution:

The discriminant $\Delta = B^2 - 4AC = -4x^2 < 0$. Hence, the equation is elliptic.

Solving the characteristic equation

$$\lambda_1 = \frac{B - i\sqrt{4AC - B^2}}{2A} = -ix \ , \quad \lambda_1 = \frac{B - i\sqrt{4AC - B^2}}{2A} = ix$$

we have

$$\frac{dy}{dx} = -ix \ , \quad \frac{dy}{dx} = ix$$

Elliptic equations

and the complex characteristic curves

$$y = -i\frac{x^2}{2} + c_1, \quad y = i\frac{x^2}{2} + c_1,$$

Taking the real and imaginary parts, the change of variables is:

$$\xi = y, \quad \eta = \frac{x^2}{2}$$

and the canonical form:

$$\frac{\partial^2 w}{\partial \xi^2} + \frac{\partial^2 w}{\partial \eta^2} = -\frac{1}{2\eta}\frac{\partial w}{\partial \eta}$$

Elliptic equations

Exercises

Transform into the canonical form the equations:

1) $u_{xx} + 6u_{xy} + 9u_{yy} = 0.$
2) $u_{xx} - 5u_{xy} + 6u_{yy} = 0.$
3) $u_{xx} - 4u_{xy} + 5u_{yy} = 0.$

Variables separation. One dimensional problems

As an example, we consider the heat equation where it is assumed that there are no heat sources,

$$\frac{\partial T}{\partial t} = a^2 \frac{\partial^2 T}{\partial x^2} \ , \quad 0 < x < l \ , \quad t > 0 \ ,$$

with homogeneous boundary conditions

$$T(0,t) = 0 \ , \quad T(l,t) = 0 \ ,$$

and the initial condition

$$T(x,0) = g(x) \ .$$

One dimensional problems

The variables separation method assumes that

$$T(x,t) = X(x)P(t)\ .$$

Substituting this solution into the equation

$$\frac{P'(t)}{a^2P(t)} = \frac{X''(x)}{X(x)} = -\lambda\ ,$$

this is,

$$P'(t) + a^2\lambda P(t) = 0\ ,$$
$$X'' + \lambda X(x) = 0\ .$$

$$X(x) = C_1\cos(x) + C_2\sin(x)$$

From the boundary conditions

$$X(0) = 0 \ , \quad X(l) = 0 \ .$$

we obtain the eigenvalues

$$\lambda_n = \left(\frac{n\pi}{l}\right)^2 \ , n = 1, 2, \ldots \ ,$$

and the eigenfunctions

$$X_n(x) = \sin\left(\frac{n\pi x}{l}\right) \ .$$

The temporal part equation is

$$P_n' + a^2 \lambda_n P_n = 0 \ ,$$

whose solution is of the form

$$P_n(t) = a_n e^{-a^2 \lambda_n t} = a_n e^{-\left(\frac{n\pi a}{l}\right)^2 t} \ .$$

One dimensional problems

The solution of the problem can be written as

$$T(x,t) = \sum_{n=1}^{\infty} a_n e^{-\left(\frac{n\pi a}{l}\right)^2 t} \sin\left(\frac{n\pi x}{l}\right) \ .$$

Since the initial condition has to be satisfied

$$g(x) = \sum_{n=1}^{\infty} a_n \sin\left(\frac{n\pi x}{l}\right) \ ,$$

and using the orthogonality property

$$a_n = \frac{2}{l}\int_0^l g(x) \sin\left(\frac{n\pi x}{l}\right) \, dx \ .$$

Exercise

Diffusion equation

We study the heat conduction in a bar. In particular, we have a bar with an initial temperature of 0 ℃ and we apply temperatures of 100 ℃ at both extremes of the bar. We assume that the length of the bar is 1 and the thermal diffusivity is α. Compute the analytical solution of this problem.

Exercise

Solution

We look for solutions of the form

$$T(x,t) = ax + b + u(x,t)\ , \quad \text{with} \quad u(0,t) = u(1,t) = 0$$

In this way

$$\begin{aligned} T(0,t) = 100 \ &\rightarrow \ b = 100 \\ T(1,t) = 100 \ &\rightarrow \ a = 0 \end{aligned}$$

Thus,

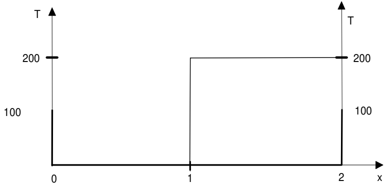
$$T(x,t) = 100 + u(x,t) = 100 + \sum_{n=1}^{\infty} A_n \sin(n\pi x)\, e^{(-\alpha n^2 \pi^2 t)}$$

Exercise

The initial condition

$$T(x,0) = g(x) = \sum_{n=1}^{\infty} A_n \sin(n\pi x)$$

where, $g(x)$ is taken as the Fourier series associated with the periodic extension of the function



Exercise

Using the orthogonality relation

$$\int_0^2 \sin(n\pi x)\sin(m\pi x)\,dx = \delta_{m,n}$$

we have

$$\begin{aligned} A_m &= \int_0^2 g(x)\sin(m\pi x)\ ,dx \\ &= \int_1^2 200\sin(m\pi x)\,dx = \frac{200}{\pi m}\left((-1)^m - 1\right) \end{aligned}$$

Multidimensional problems

Example

We consider the Laplace equation

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = 0 \ ,$$

on a rectangle of edges (a, b, c), assuming the boundary conditions

$$u(0, y, z) = u(a, y, z) = u(x, 0, z) = u(x, b, z) = u(x, y, 0) = 0 \ ,$$

and $u(x, y, c) = V(x, y)$.

Multidimensional problems

Using the variables separation method,

$$u(x,y,z) = X(x)Y(y)Z(z) \ .$$

we obtain the equation

$$\frac{1}{X(x)}\frac{d^2X}{dx^2} + \frac{1}{Y(y)}\frac{d^2Y}{dy^2} + \frac{1}{Z(z)}\frac{d^2Z}{dz^2} = 0 \ .$$

Assuming that

$$\frac{1}{X(x)}\frac{d^2X}{dx^2} = -\alpha^2$$
$$\frac{1}{Y(y)}\frac{d^2Y}{dy^2} = -\beta^2$$
$$\frac{1}{Z(z)}\frac{d^2Z}{dz^2} = \gamma^2$$

with $\gamma^2 = \alpha^2 + \beta^2$.

Multidimensional problems

we have that $X(x)$, $Y(y)$ and $Z(z)$ must satisfy

$$X(0) = X(a) = Y(0) = Y(b) = Z(0) = 0 \ .$$

Hence, we obtain the solutions

$$\begin{aligned} X(x) &= \sin(\alpha x) \\ Y(y) &= \sin(\beta y) \\ Z(z) &= \sinh\left(\sqrt{\alpha^2+\beta^2} z\right) \end{aligned}$$

being

$$\alpha = \frac{\pi n}{a} \ , \quad \beta = \frac{\pi m}{b} \ , \quad n, m \in \mathbb{Z} \ .$$

Multidimensional problems

The general form of the solution is

$$u(x,y,z)=\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}A_{m,n}\sin\left(\alpha_n x\right)\sin\left(\beta_m y\right)\sinh\left(\gamma_{n,m}z\right)$$

with

$$\alpha_n=\frac{\pi n}{a}\ ,\quad \beta_m=\frac{\pi m}{b}\ ,\quad \gamma_{n,m}=\pi\sqrt{\frac{n^2}{a^2}+\frac{m^2}{b^2}}\ .$$

Making $u(x,y,c)=V(x,y)$, we obtain

$$\boxed{V(x,y)=\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}A_{m,n}\sin\left(\alpha_n x\right)\sin\left(\beta_m y\right)\sinh\left(\gamma_{n,m}c\right)\ ,}$$

with

$$A_{n,m}=\frac{4}{ab\sinh\left(\gamma_{n,m}c\right)}\int_0^a dx\int_0^b dy V(x,y)\sin\left(\alpha_n x\right)\sin\left(\beta_m y\right)\ .$$