UNIVERSITAT POLITÈCNICA DE VALÈNCIA

Fluids equations

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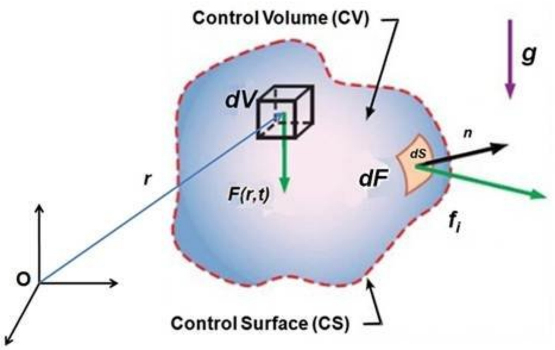
Outline

Conservation principles

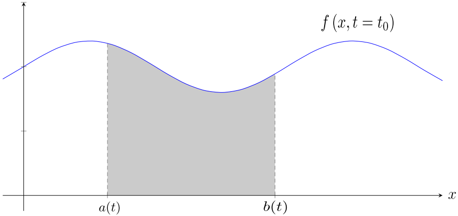
- Normally, two main types of fluids appear in technical applications: water and air.
- The general technique for obtaining the equations that describe the motion of a fluid is to consider a small control volume, V, through which the fluid moves. (Eulerian description).
- These equations will be obtained by applying conservation of mass, Newton's Second Law and conservation of energy to the control volume.

Continuity equation



Reynolds' theorem

First, we consider a 1D problem



For a given function $f(x)$, the definite integral

$$M = \int_{a(t)}^{b(t)} f(x,t)\, dx$$

satisfies

$$\frac{dM}{dt} = \frac{\partial M}{\partial t} + \frac{\partial M}{\partial a}\frac{\partial a}{\partial t} + \frac{\partial M}{\partial b}\frac{\partial b}{\partial t}$$

Reynolds' theorem

That is,

$$\frac{dM}{dt} = \int_{a(t)}^{b(t)} \frac{\partial f(x,t)}{\partial t}\,dx + f(b(t),t)\frac{db}{dt} - f(a(t),t)\frac{da}{dt}$$

A similar result can be obtained for a problem en 3D, which is known as the Reynolds' transport theorem.

Continuity equation

For a magnitude $N = \eta\rho$ in a control volume V we have that the Reynold's theorem states that

$$\frac{dN}{dt} = \int_V \frac{\partial}{\partial t}(\eta\rho)\, dV + \int_S \eta\rho\vec{v}\, d\vec{S}\ .$$

The mass conservation

$$\frac{dM}{dt} = \int_V \frac{\partial\rho}{\partial t}\, dV + \int_S \rho\vec{v}\, d\vec{S} = 0\ .$$

Continuity equation

The variation of the mass with respect to time inside the control volume is equal to the mass flow rate through the surface of the control volume,

$$\int_V \frac{\partial \rho}{\partial t} \, dV = - \int_S \rho \, \vec{v} \, \vec{n} \, dS \ ,$$

Using Gauss' Theorem, we have

$$\int_V \left(\frac{\partial \rho}{\partial t} + \vec{\nabla} \left(\rho \vec{v} \right) \right) \, dV = 0 \ .$$

Continuity equation

The continuity equation

$$\frac{\partial \rho}{\partial t} + \vec{\nabla}\left(\rho \vec{v}\right) = 0 \ ,$$

$$\frac{D\rho}{Dt} + \rho \vec{\nabla} \vec{v} = 0 \ ,$$

Continuity equation

The material derivative:

$$\frac{D\rho}{Dt} = \frac{\partial \rho}{\partial t} + v_x \frac{\partial \rho}{\partial x} + v_y \frac{\partial \rho}{\partial y} + v_z \frac{\partial \rho}{\partial z} = \frac{\partial \rho}{\partial t} + v_i \frac{\partial \rho}{\partial x_i} \ ,$$

and the divergence

$$\vec{\nabla} \vec{v} = \frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} + \frac{\partial v_z}{\partial z} = \frac{\partial v_i}{\partial x_i} \ .$$

Momentum equation

The momentum equation is obtained by applying Newton's Second Law to the control volume V.

$$\sum_l \vec{F}_l = \frac{d\vec{P}}{dt} \ .$$

The variation of the momentun, $\vec{P}$, with respect to time in the volume V, is given by

$$\frac{d\vec{P}}{dt} = \int_V \frac{\partial(\rho\,\vec{v})}{\partial t}\,dV + \int_S \rho\,\vec{v}\,(\vec{v}\,\vec{n})\,dS \ .$$

Using Gauss' theorem, in components, we can write

$$
\begin{aligned}
F_x &= \int_V \frac{\partial}{\partial t}(\rho\, v_x)\ dV + \int_V \vec{\nabla}\,(\rho v_x \vec{v})\ dV\ , \\
F_y &= \int_V \frac{\partial}{\partial t}(\rho\, v_y)\ dV + \int_V \vec{\nabla}\,(\rho v_y \vec{v})\ dV\ , \\
F_z &= \int_V \frac{\partial}{\partial t}(\rho\, v_z)\ dV + \int_V \vec{\nabla}\,(\rho v_z \vec{v})\ dV\ .
\end{aligned}
$$

Momentum equation

For the component x, it is satisfied

$$\frac{\partial}{\partial t}\left(\rho\, v_x\right) + \vec{\nabla}\left(\rho v_x \vec{v}\right) =$$
$$\rho \frac{\partial v_x}{\partial t} + v_x \left(\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x}\left(\rho v_x\right) + \frac{\partial \rho}{\partial y} v_y + \rho \frac{\partial v_y}{\partial y} + \frac{\partial \rho}{\partial z} v_z + \rho \frac{\partial v_z}{\partial z} \right) +$$
$$\rho v_x \frac{\partial v_x}{\partial x} + \rho v_y \frac{\partial v_x}{\partial y} + \rho v_z \frac{\partial v_x}{\partial z} \ .$$

Using the continuity equation,

$$\frac{\partial}{\partial t}\left(\rho\, v_x\right) + \vec{\nabla}\left(\rho v_x \vec{v}\right) =$$
$$\rho \left(\frac{\partial v_x}{\partial t} + v_x \frac{\partial v_x}{\partial x} + v_y \frac{\partial v_x}{\partial y} + v_z \frac{\partial v_x}{\partial z} \right) \ .$$

Momentum equation

The momentum equation

$$\int_V \rho \left(\frac{\partial \vec{v}}{\partial t} + \left(\vec{v}\, \vec{\nabla} \right) \vec{v} \right) dV = \int_V \rho \frac{D\vec{v}}{Dt}\, dV = \vec{F} \ .$$

If we consider a non-viscous fluid, the only forces acting on the control volume are the force of gravity and the force exerted by the pressure, p,

$$\vec{F} = \int_V \rho \vec{g}\, dV - \int_S p \vec{n}\, dS = \int_V \left(\rho \vec{g} - \vec{\nabla} p \right) dV \ .$$

Momentum equation

The Euler's equations

$$\rho \frac{D\vec{v}}{Dt} = \rho \vec{g} - \vec{\nabla} p$$

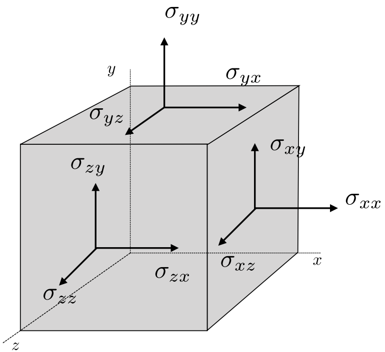
If the reference frame is chosen in such a way that gravity is directed in the direction of the negative z-axis, the Euler equations in components are,

$$\rho \left(\frac{\partial v_x}{\partial t} + v_x \frac{\partial v_x}{\partial x} + v_y \frac{\partial v_x}{\partial y} + v_z \frac{\partial v_x}{\partial z} \right) = -\frac{\partial p}{\partial x}$$
$$\rho \left(\frac{\partial v_y}{\partial t} + v_x \frac{\partial v_y}{\partial x} + v_y \frac{\partial v_y}{\partial y} + v_z \frac{\partial v_y}{\partial z} \right) = -\frac{\partial p}{\partial y}$$
$$\rho \left(\frac{\partial v_z}{\partial t} + v_x \frac{\partial v_z}{\partial x} + v_y \frac{\partial v_z}{\partial y} + v_z \frac{\partial v_z}{\partial z} \right) = -\rho g - \frac{\partial p}{\partial z} \ .$$

Momentum equation

When a viscous fluid is considered, it is necessary to introduce the tensor that accounts for the stresses within the fluid,

$$\sigma = \begin{pmatrix} \sigma_{xx} & \sigma_{xy} & \sigma_{xz} \\ \sigma_{yx} & \sigma_{yy} & \sigma_{yz} \\ \sigma_{zx} & \sigma_{zy} & \sigma_{zz} \end{pmatrix} ,$$



Momentum equation

The momentum equations are written as

$$\rho \frac{D\vec{v}}{Dt} = \rho \vec{g} - \vec{\nabla} p + \vec{\nabla} \sigma \ ,$$

In cartesian coordinates,

$$\begin{aligned}
\rho \frac{Dv_x}{Dt} &= \rho g_x - \frac{\partial p}{\partial x} + \frac{\partial \sigma_{xx}}{\partial x} + \frac{\partial \sigma_{yx}}{\partial y} + \frac{\partial \sigma_{zx}}{\partial z} \ , \\
\rho \frac{Dv_y}{Dt} &= \rho g_y - \frac{\partial p}{\partial y} + \frac{\partial \sigma_{xy}}{\partial x} + \frac{\partial \sigma_{yy}}{\partial y} + \frac{\partial \sigma_{zy}}{\partial z} \ , \\
\rho \frac{Dv_z}{Dt} &= \rho g_z - \frac{\partial p}{\partial z} + \frac{\partial \sigma_{xz}}{\partial x} + \frac{\partial \sigma_{yz}}{\partial y} + \frac{\partial \sigma_{zz}}{\partial z} \ ,
\end{aligned}$$

Momentum equation

Newton observed that the stress tensor in a fluid is proportional to the partial derivatives of the the velocities.
For Newtonian fluids it is satisfied that

$$
\begin{aligned}
\sigma_{xx} &= \lambda\left(\vec{\nabla}\vec{v}\right) + 2\mu\frac{\partial v_x}{\partial x}\ , \quad \sigma_{yy} = \lambda\left(\vec{\nabla}\vec{v}\right) + 2\mu\frac{\partial v_y}{\partial y}\ , \\
\sigma_{zz} &= \lambda\left(\vec{\nabla}\vec{v}\right) + 2\mu\frac{\partial v_z}{\partial z}\ , \\
\sigma_{xy} &= \sigma_{yx} = \mu\left(\frac{\partial v_y}{\partial x} + \frac{\partial v_x}{\partial y}\right)\ , \\
\sigma_{xz} &= \sigma_{zx} = \mu\left(\frac{\partial v_x}{\partial z} + \frac{\partial v_z}{\partial x}\right)\ , \\
\sigma_{yz} &= \sigma_{zy} = \mu\left(\frac{\partial v_z}{\partial y} + \frac{\partial v_y}{\partial z}\right)\ ,
\end{aligned}
$$

where μ is the molecular viscosity coefficient and λ is the second viscosity coefficient.

Momentum equation

It is assumed that

$$\lambda = -\frac{2}{3}\mu \ ,$$

The Navier-Stokes' equations for a viscous fluid

$$\begin{aligned}
\rho\frac{Dv_x}{Dt} &= \rho g_x - \frac{\partial p}{\partial x} - \frac{2}{3}\frac{\partial\left(\mu\vec{\nabla}\vec{v}\right)}{\partial x} + 2\frac{\partial}{\partial x}\left(\mu\frac{\partial v_x}{\partial x}\right) \\
&+\frac{\partial}{\partial y}\left(\mu\left(\frac{\partial v_x}{\partial y} + \frac{\partial v_y}{\partial x}\right)\right) + \frac{\partial}{\partial z}\left(\mu\left(\frac{\partial v_x}{\partial z} + \frac{\partial v_z}{\partial x}\right)\right) \ , \\
\rho\frac{Dv_y}{Dt} &= \rho g_y - \frac{\partial p}{\partial y} - \frac{2}{3}\frac{\partial\left(\mu\vec{\nabla}\vec{v}\right)}{\partial y} + \frac{\partial}{\partial x}\left(\mu\left(\frac{\partial v_x}{\partial y} + \frac{\partial v_y}{\partial x}\right)\right) \\
&+2\frac{\partial}{\partial y}\left(\mu\frac{\partial v_y}{\partial y}\right) + \frac{\partial}{\partial z}\left(\mu\left(\frac{\partial v_x}{\partial z} + \frac{\partial v_z}{\partial x}\right)\right) \ , \\
\rho\frac{Dv_z}{Dt} &= \rho g_z - \frac{\partial p}{\partial z} - \frac{2}{3}\frac{\partial\left(\mu\vec{\nabla}\vec{v}\right)}{\partial z} + \frac{\partial}{\partial x}\left(\mu\left(\frac{\partial v_x}{\partial z} + \frac{\partial v_z}{\partial x}\right)\right) \\
&+\frac{\partial}{\partial y}\left(\mu\left(\frac{\partial v_y}{\partial z} + \frac{\partial v_z}{\partial y}\right)\right) + 2\frac{\partial}{\partial z}\left(\mu\frac{\partial v_z}{\partial z}\right) \ .
\end{aligned}$$

Momentum equation

For a 2-D incompressible flow with constant density and viscosity with negligible body forces, the Navier-Stokes equations can be greatly simplified,

$$
\begin{aligned}
\frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} &= 0 \\
\frac{\partial v_x}{\partial t} + \frac{\partial (v_x)^2}{\partial x} + \frac{\partial v_x v_y}{\partial y} &= \nu \left(\frac{\partial^2 v_x}{\partial x^2} + \frac{\partial^2 v_x}{\partial y^2} \right) - \frac{1}{\rho} \frac{\partial p}{\partial x} \\
\frac{\partial v_y}{\partial t} + \frac{\partial v_x v_y}{\partial x} + \frac{\partial (v_y)^2}{\partial y} &= \nu \left(\frac{\partial^2 v_y}{\partial x^2} + \frac{\partial^2 v_y}{\partial y^2} \right) - \frac{1}{\rho} \frac{\partial p}{\partial y}
\end{aligned}
$$

where $\nu = \mu/\rho$ is the kinematic viscosity.

Momentum equation

Exercise

Obtain the non-conservative form of the Navier-stokes equations exposed above:

$$
\begin{aligned}
\frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} &= 0 \\
\frac{\partial v_x}{\partial t} + v_x \frac{\partial v_x}{\partial x} + v_y \frac{\partial v_x}{\partial y} &= \nu \left(\frac{\partial^2 v_x}{\partial x^2} + \frac{\partial^2 v_x}{\partial y^2} \right) - \frac{1}{\rho} \frac{\partial p}{\partial x} \\
\frac{\partial v_y}{\partial t} + v_x \frac{\partial v_y}{\partial x} + v_y \frac{\partial v_y}{\partial y} &= \nu \left(\frac{\partial^2 v_y}{\partial x^2} + \frac{\partial^2 v_y}{\partial y^2} \right) - \frac{1}{\rho} \frac{\partial p}{\partial y}
\end{aligned}
$$

Momentum equation

Exercise

Starting from the equation of motion

$$\frac{\partial \vec{v}}{\partial t} + \left(\vec{v}\vec{\nabla}\right)\vec{v} = -\frac{1}{\rho}\vec{\nabla}p + \nu\nabla^2\vec{v}$$

obtain the equations associated with the vorticity, defined as $\vec{\omega} = \vec{\nabla} \times \vec{v}$ for a 2D problem defined in the (x, y) plane.

Momentum equation

Vorticity

$$\vec{\omega} = \vec{\nabla} \times \vec{v}$$

Is a measure of the local rotation of the fluid

$$\int_S \vec{\omega}\, d\vec{S} = \int_S \left(\vec{\nabla} \times \vec{v}\right)\, d\vec{S} = \int_\sigma \vec{v}\, d\vec{r}$$

cyclonic vortex in the atmosphere

The vorticity equation

$$\frac{\partial \vec{\omega}}{\partial t} + \left(\vec{v}\vec{\nabla}\right)\vec{\omega} = \left(\vec{\omega}\vec{\nabla}\right)\vec{v} + \nu \nabla^2 \vec{\omega}$$

Energy equation

Consider a fixed volume V surrounded by a surface S.
The total energy content of the fluid contained within V is

$$E = \int_V \rho\, e\, dV + \int_V \frac{1}{2}\, \rho\, v_i\, v_i\, dV \ ,$$

Here, e is the internal (i.e., thermal) energy per mass unit of the fluid.
The energy flux across S , and out of V , is

$$\Phi_E = \int_S \rho \left(e + \frac{1}{2}\, v_i\, v_i \right) v_j\, dS_j = \int_V \frac{\partial}{\partial x_j} \left(\rho \left(e + \frac{1}{2}\, v_i\, v_i \right) v_j \right) dV \ ,$$

where use has been made of the Gauss' theorem.

Energy equation

According to the first law of thermodynamics, the rate of increase of the energy contained within V, plus the net energy flux out of V, is equal to the net rate of work done on the fluid within V, minus the net heat flux out of V :

$$\frac{dE}{dt} + \Phi_E = \dot{W} - \dot{Q} \ ,$$

The net rate of work due to volumetric forces is

$$\int_V \rho \vec{f} \vec{v} \, dV \ .$$

and the rate of work associated to pressure

$$-\int_S p v_i dS_i = -\int_V \frac{\partial}{\partial x_i} \left(v_i p \right) \, dV \ .$$

Energy equation

The rate of work done by the stress tensor is

$$\int_S v_i \sigma_{ij} dS_j = \int_V \frac{\partial}{\partial x_j} \left(v_i \sigma_{ij} \right) \, dV \ .$$

Thus,

$$\boxed{\dot{W} = \int_V \rho \vec{f} \vec{v} \, dV - \int_V \frac{\partial}{\partial x_i} \left(v_i p \right) \, dV + \int_V \frac{\partial}{\partial x_j} \left(v_i \sigma_{ij} \right) \, dV \ .}$$

Energy equation

- The heat flow in fluids is driven by temperature gradients.
- Let the q_i be the Cartesian components of the heat flux density. It follows that the heat flux across a surface element dS, is $\vec{q}d\vec{S} = q_i\, dS_i$.
- Let T be the temperature of the fluid. We assume that the Fourier's Law is valid,

$$q_i = -k\frac{\partial T}{\partial x_i}\ .$$

- The neat heat flux out a volume V is,

$$\dot{Q} = -\int_S k\frac{\partial T}{\partial x_i}\, dS_i = -\int_V \frac{\partial}{\partial x_i}\left(k\frac{\partial T}{\partial x_i}\right)\, dV\ .$$

Energy equation

The energy conservation equation,

$$\int_V \left(\frac{\partial}{\partial t} \left(\rho \left(e + \frac{1}{2} v_i v_i \right) \right) + \frac{\partial}{\partial x_j} \left(\rho \left(e + \frac{1}{2} v_i v_i \right) v_j \right) \right) dV$$
$$= \int_V \left(\rho f_i v_i - \frac{\partial}{\partial x_j} \left(v_j p \right) + \frac{\partial}{\partial x_j} \left(v_i \sigma_{ij} + k \frac{\partial T}{\partial x_j} \right) \right) dV \ .$$

Energy equation

Using the continuity equation

$$\rho \frac{D}{Dt}\left(e + \frac{1}{2}v_i v_i\right) = \rho f_i v_i - \frac{\partial}{\partial x_j}\left(v_j p\right) + \frac{\partial}{\partial x_j}\left(v_i \sigma_{ij} + k\frac{\partial T}{\partial x_j}\right) \ ,$$

The momentum equation can be written as,

$$\rho v_i \frac{Dv_i}{Dt} = \rho \frac{D}{Dt}\left(\frac{1}{2}v_i v_i\right) = -v_i \frac{\partial p}{\partial x_i} + v_i \rho f_i + v_i \frac{\partial \sigma_{ij}}{\partial x_j} \ .$$

Combining these equations the energy equation,

$$\boxed{\rho \frac{De}{Dt} = \frac{\partial v_i}{\partial x_j}\sigma_{ij} + \frac{\partial}{\partial x_j}\left(k\frac{\partial T}{\partial x_j}\right) \ .}$$

Energy equation

The stress tensor in an isotropic Newtonian fluid is

$$\sigma_{ij} = -\lambda\,\delta_{ij} + 2\,\mu\left(e_{ij} - \frac{1}{3}\,e_{kk}\,\delta_{ij}\right),$$

where

$$e_{ij} = \frac{1}{2}\left(\frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i}\right) ,$$

and λ and μ are scalars.
The energy conservation equation for an isotropic Newtonian fluid takes the general form

$$\frac{De}{Dt} = -\frac{p}{\rho}\,\frac{\partial v_i}{\partial x_i} + \frac{1}{\rho}\left(\chi + \frac{\partial}{\partial x_j}\left(k\,\frac{\partial T}{\partial x_j}\right)\right).$$

Energy equation

where,

$$\chi = \mu \left(\frac{\partial v_i}{\partial x_j} \frac{\partial v_i}{\partial x_j} + \frac{\partial v_i}{\partial x_j} \frac{\partial v_j}{\partial x_i} - \frac{2}{3} \frac{\partial v_i}{\partial x_i} \frac{\partial v_j}{\partial x_j} \right) \;,$$

is the rate of heat generation per unit volume due to viscosity.
In vector notation the energy equation is writen as

$$\boxed{\frac{De}{Dt} = -\frac{p}{\rho} \vec{\nabla} \vec{v} + \frac{\chi}{\rho} + \frac{\vec{\nabla}(k \, \vec{\nabla} T)}{\rho} \;.}$$

Energy equation

- The equations for the mass, momentum and energy are 5 equations whose unknowns are $\rho, p, v_x, v_y, v_z, e$ and T. Hence, we need more relations between the unknowns.
- In many situations of general interest, the flow of gases is compressible. We can use the thermodynamic relations that specify the internal energy per unit mass, and the temperature in terms of the density and pressure.
  For an ideal gas, these relations can take the form:

$$\boxed{e = \frac{c_v}{M} T, \quad T = \frac{M}{R} \frac{p}{\rho}}$$

  where c_v is the molar specific heat at constant volume, $R = 8.3145\,\mathrm{J\,K^{-1}\,mol^{-1}}$ the molar ideal gas constant, M the molar mass (i.e., the mass of 1 mole of gas molecules), and T the temperature in degrees Kelvin.

Energy equation

The energy equation can be also formulated in terms of the enthalpy

$$\rho \frac{Dh}{Dt} - \frac{Dp}{Dt} = \Phi + \frac{\partial}{\partial x}\left(k\frac{\partial T}{\partial x}\right) + \frac{\partial}{\partial y}\left(k\frac{\partial T}{\partial y}\right) + \frac{\partial}{\partial z}\left(k\frac{\partial T}{\partial z}\right)$$

where

$$h = c_p \left(T - T_{\text{ref}}\right)$$

Conservation of scalar quantities

- The integral form of the equation describing conservation of a scalar quantity, ϕ, is

$$\frac{\partial}{\partial t}\int_V \rho\phi \, dV + \int_S \rho\phi\vec{v}\, d\vec{S} = \sum f_\phi$$

  where f_ϕ represents transport of ϕ by mechanisms other than convection and any sources or sinks of the scalar.

- Diffusion transport is always present and it is usually described by a gradient approximation, e.g.,

$$f_\phi^d = \int_S \Gamma\vec{\nabla}\phi \, d\vec{S},$$

  where Γ is the diffusivity for the quantity ϕ.

Conservation of scalar quantities

An example is the energy equation for a fluid with constant specific heat that can be expressed as

$$\frac{\partial}{\partial t}\int_V \rho T\, dV + \int_S \rho T \vec{v}\, d\vec{S} = \int_S \frac{\nu}{Pr}\vec{\nabla} T\, d\vec{S},$$

where the thermal conductivity is assumed to be $k = \nu c_p / Pr$, where Pr is the Prandtl number, which is defined as the ratio of momentum diffusivity to heat diffusivity,

$$P_r = \frac{\nu}{\alpha} = \frac{\text{momentum diffusivity}}{\text{thermal diffusivity}} = \frac{\nu\rho}{k/(c_p\rho)} = \frac{c_p\mu}{k}$$

c_p is the specific heat at constant pressure, μ the dynamical viscosity and k the thermal conductivity.

Conservation of scalar quantities

The integral form of the generic conservation equation is

$$\frac{\partial}{\partial t}\int_V \rho\phi\, dV + \int_S \rho\phi\vec{v}\, d\vec{S} = \int_S \Gamma\vec{\nabla}\phi d\vec{S} + \int_V q_\phi dV,$$

where q_ϕ is the source or sink of ϕ.

The coordinate-free vector form of this equation is,

$$\frac{\partial(\rho\phi)}{\partial t} + \vec{\nabla}\left(\rho\phi\vec{v}\right) = \vec{\nabla}\left(\Gamma\vec{\nabla}\phi\right) + q_\phi.$$

Conservation of scalar quantities

In Cartesian coordinates and tensor notation, the differential form of the generic conservation equation is:

$$\boxed{\frac{\partial(\rho\phi)}{\partial t} + \frac{\partial\left(\rho\phi v_j\right)}{\partial x_j} = \frac{\partial}{\partial x_j}\left(\Gamma\frac{\partial\phi}{\partial x_j}\right) + q_\phi.}$$

Dimensionless Form of Equations

- Experimental studies of flows are often carried out on models, and the results are displayed in dimensionless form, thus allowing scaling to real flow conditions. The same approach can be undertaken in numerical studies as well.
- The governing equations can be transformed to dimensionless form by using appropriate normalization.
- Velocities can be normalized by a reference velocity v_0 , spatial coordinates by a reference length L_0, time by some reference time t_0, pressure by ρv_0^2, and temperature by some reference temperature difference $(T_1 - T_0)$.

Dimensionless Form of Equations

The dimensionless variables are then:

$$t^* = \frac{t}{t_0},\ x_i^* = \frac{x_i}{L_0},\ v_i^* = \frac{v_i}{v_0},\ p^* = \frac{p}{\rho v_0^2},\ T^* = \frac{T - T_0}{T_1 - T_0}.$$

Dimensionless Form of Equations

If the fluid properties are constant, the continuity, momentum and energy equations are, in dimensionless form:

$$
\begin{aligned}
\frac{\partial v_i^*}{\partial x_i^*} &= 0, \\
St\frac{v_i^*}{\partial t^*} + \frac{\partial\left(v_j^* v_i^*\right)}{\partial x_j^*} &= \frac{1}{Re}\frac{\partial^2 v_i^*}{\partial x_j^{*2}} - \frac{\partial p^*}{\partial x_i^*} + \frac{1}{Fr^2}\gamma_i, \\
St\frac{\partial T^*}{\partial t^*} + \frac{\partial\left(v_j^* T^*\right)}{\partial x_j^*} &= \frac{1}{RePr}\,\frac{\partial^2 T^*}{\partial x_j^{*2}}.
\end{aligned}
$$

Dimensionless Form of Equations

- The following dimensionless numbers appear in the equations:

$$St = \frac{L_0}{v_0 t_0},\ \ Re = \frac{\rho v_0 L_0}{\mu},\ \ Fr = \frac{v_0}{\sqrt{L_0 g}},$$

  which are called Strouhal, Reynolds, and Froude number, respectively. γ_i is the component of the normalized gravitational acceleration vector in the x_i direction.
- The choice of the normalization quantities is obvious in simple flows; v_0 is the mean velocity and L_0 is a geometric length scale; T_0 and T_1 are the cold and hot wall temperatures.
- If the geometry is complicated, the fluid properties are not constant, or the boundary conditions are unsteady, the number of dimensionless parameters needed to describe a flow can become very large and dimensionless equations may no longer be useful.

Dimensionless Form of Equations

Example

Calculating the two-dimensional flow around a cylinder (radius a, located at $x = y = 0$) in a uniform stream U involves solving

$$\frac{\partial \vec{v}}{\partial t} + \left(\vec{v}\vec{\nabla}\right)\vec{v} = -\frac{1}{\rho}\vec{\nabla}p + \nu\nabla^2\vec{v}, \quad \vec{\nabla}\vec{v} = 0 \ ,$$

with boundary conditions

$$\vec{v} = \vec{0} \ \text{ on } \ x^2 + y^2 = a^2, \quad \vec{v} \to (U, 0) \text{ as } x^2 + y^2 \to \infty$$

Rewrite this problem in non-dimensional form using the dimensionless variables

$$\vec{x}' = \vec{x}/a, \ \vec{v}' = \vec{v}/U, \ p' = p/\rho U^2, \ t' = tU/a \ .$$

Dimensionless Form of Equations

Note that $\vec{x}' = \vec{x}/a$ implies $\vec{\nabla}' = a\vec{\nabla}$ and $t' = tU/a$ gives $\frac{\partial}{\partial t} = \frac{U}{a}\frac{\partial}{\partial t'}$.

The equation

$$\frac{U^2}{a}\frac{\partial \vec{v}'}{\partial t'} + \frac{U^2}{a}\left(\vec{v}'\vec{\nabla}'\right)\vec{v}' = -\frac{\rho U^2}{\rho a}\vec{\nabla}' p' + \frac{\nu U}{a^2}\nabla'^2\vec{v}'$$

That is

$$\frac{\partial \vec{v}'}{\partial t'} + \left(\vec{v}'\vec{\nabla}'\right)\vec{v}' = -\vec{\nabla}' p' + \frac{1}{\mathrm{Re}}\nabla'^2\vec{v}'$$

where $\mathrm{Re} = \frac{\frac{U^2}{a}}{\frac{\nu U}{a^2}} = \frac{\text{inertial forces}}{\text{viscous forces}}$

Dimensionless Form of Equations

Exercise

The differential equation describing the movement in the $x - y$ plane of a non viscous fluid is

$$\frac{\partial^2 \phi}{\partial t^2} + \frac{\partial}{\partial t}\left(u^2 + v^2\right) + \left(u^2 - a^2\right)\frac{\partial^2 \phi}{\partial x^2} + \left(v^2 - a^2\right)\frac{\partial^2 \phi}{\partial y^2} + 2u\,v\frac{\partial^2 \phi}{\partial x \partial y} = 0,$$

where ϕ is the velocity potential and a is the sound velocity. Write the corresponding dimensionless equation using the parameters L for a typical length and a_0 for the sound velocity at the entrance to define the dimensionless variables.

Simplified Mathematical Models

- The conservation equations for mass and momentum are non-linear, coupled, and very difficult to solve.
- Only in a small number of cases-mostly fully developed flows in simple geometries, e.g., in pipes, between parallel plates etc. it is possible to obtain an analytical solution of the Navier-Stokes equations, but their practical relevance is limited.
- In most cases, even the simplified equations cannot be solved analytically; one has to use numerical methods, but the computing effort may be much smaller than for the full equations.

Incompressible flow

- In most situations, the flow of a conventional liquid, such as water, is incompressible to a high degree of accuracy. (Also air when Mach number (v/a) is smaller than 0.3).
- For an incompressible fluid, the rate of change of ρ following the motion is zero: that is,

$$\frac{D\rho}{Dt} = 0.$$

- The continuity equation reduces to

$$\boxed{\vec{\nabla}\vec{v} = 0 \ .}$$

In this case, that is, an incompressible fluid must have a divergence-free, or solenoidal, velocity field.

- Suppose that the volume force acting on the fluid is conservative, that is,

$$\vec{F} = -\rho\,\vec{\nabla}\vec{\Psi}$$

Ψ is the potential energy per unit mass.

- Assuming that the fluid viscosity is a spatially uniform quantity, which is generally the case (unless there are strong temperature variations within the fluid), the Navier-Stokes equation for an incompressible fluid reduces to

$$\frac{D\vec{v}}{Dt} = -\frac{\vec{\nabla}p}{\rho} - \vec{\nabla}\Psi + \nu\,\nabla^2\vec{v},$$

where

$$\nu = \frac{\mu}{\rho}\ .$$

is termed the kinematic viscosity, and has units of m^2/s.

Incompressible flow

- The complete set of equations governing incompressible flow is

$$\vec{\nabla}\vec{v} = 0,$$
$$\frac{D\vec{v}}{Dt} = -\frac{\vec{\nabla}p}{\rho} - \vec{\nabla}\Psi + \nu\,\nabla^2\vec{v}.$$

- Here, ρ and ν are regarded as known constants, and Ψ as a known function. Thus, we have four equations for four unknowns, the pressure, p, plus the three components of the velocity, $\vec{v}$.
- Note that an energy conservation equation is redundant in the case of incompressible fluid flow.

Inviscid (Euler) Flow

- In flows far from solid surfaces, the effects of viscosity are usually very small.
- If viscous effects are neglected altogether, i.e., if we assume that the stress tensor reduces to $\sigma = 0$, the Navier-Stokes equations reduce to the Euler equations.
- The continuity equation is

$$\boxed{\frac{\partial \rho}{\partial t} + \vec{\nabla}\left(\rho \vec{v}\right) = 0,}$$

- The momentum equations are:

$$\boxed{\frac{\partial\left(\rho v_i\right)}{\partial t} + \vec{\nabla}\left(\rho v_i \vec{v}\right) = -\vec{\nabla}\left(p \vec{e}_i\right) + \rho g_i.}$$

Potential Flow

- The fluid is assumed to be inviscid (as in the Euler equations); however, an additional condition is imposed on the flow, the velocity field must be irrotational, i.e.:

$$\vec{\nabla} \times \vec{v} = \vec{0}.$$

  From this condition it follows that there exists a velocity potential Φ, such that the velocity vector can be defined as $\vec{v} = -\vec{\nabla}\Phi$.

- The continuity equation for an incompressible flow, $\vec{\nabla}\vec{v} = 0$, then becomes a Laplace equation for the potential

$$\vec{\nabla}\left(\vec{\nabla}\Phi\right) = 0.$$

  The momentum equation can then be integrated to give the Bernoulli equation, an algebraic equation that can be solved once the potential is known.

Creeping (Stokes) Flow

- When the flow velocity is very small, the fluid is very viscous, or the geometric dimensions are very small (i.e., when the Reynolds number is small), the convection (inertial) terms in the Navier-Stokes equations can be neglected.
- If the fluid properties can be considered constant, the momentum equations become linear; they are usually called the Stokes equations. Due to the low velocities the unsteady term can also be neglected.
- The continuity equation is

$$\boxed{\vec{\nabla}\vec{v} = 0\ ,}$$

- The momentum equations become:

$$\boxed{\vec{\nabla}\left(\mu\vec{\nabla}v_i\right) - \frac{1}{\rho}\vec{\nabla}\left(p\vec{e}_i\right) + g_i = 0\ .}$$

- Creeping flows are found in porous media, coating technology, micro-devices, etc.

Fluid equations classification

- The classification of linear second order partial differential equations is based on the nature of the characteristics, curves along which information about the solution is carried.
- In the hyperbolic case, the characteristics are real and distinct. This means that information propagates at finite speeds in two sets of directions. In general, the information propagation is in a particular direction so that one datum needs to be given at an initial point on each characteristic; the two sets of characteristics therefore demand two initial conditions. If there are lateral boundaries, usually only one condition is required at each point because one characteristic is carrying information out of the domain and one is carrying information in.

Fluid equations classification

- In parabolic equations the characteristics degenerate to a single real set. Consequently, only one initial condition is normally required. At lateral boundaries one condition is needed at each point.
- Finally, in the elliptic case, there are no real characteristics; the two sets of characteristics are complex (imaginary) and distinct. As a consequence, there are no special directions of information propagation. Indeed, information travels essentially equally well in all directions.

  Generally, one boundary condition is required at each point on the boundary and the domain of solution is usually closed although part of the domain may extend to infinity.

Fluid equations classification

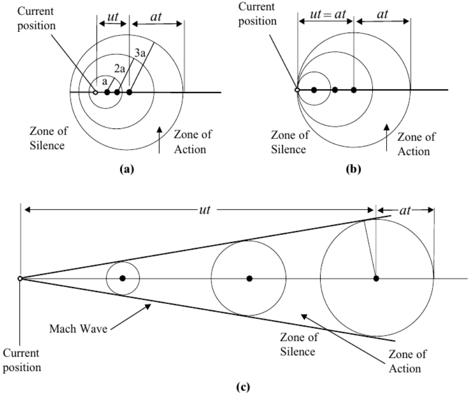
- These differences in the nature of the equations are reflected in the methods used to solve them. It is an important general rule that numerical methods should respect the properties of the equations they are solving.
- Consider that the flow velocity u is the velocity of a body moving in a fluid. The movement of this body disturbs the fluid particles ahead of the body, setting off the propagation velocity equal to the speed of sound a.

  The ratio of these two competing speeds is defined as Mach number

$$M = \frac{u}{a} \ .$$

Fluid equations classification

The physical situations these types of equations represent can be illustrated by the flow velocity relative to the speed of sound as shown in Figure



(a) Subsonic flow $(u < a, M < 1)$, (b) sonic flow $(u = a, M = 1)$, (c) supersonic flow $(u > a, M > 1)$.

Fluid equations classification

- For subsonic speed, $M < 1$, as time t increases, the body moves a distance, ut, which is always shorter than the distance at of the sound wave. The sound wave reaches the observer, prior to the arrival of the body. The zones outside and inside of the circles are known as the zone of silence and zone of action, respectively.
- If the body travels at the speed of sound, $M = 1$, then the observer does not hear the body approaching him prior to the arrival of the body. All circles representing the distance travelled by the sound wave are tangent to the vertical line at the position of the observer.
- For supersonic speed, $M > 1$, the velocity of the body is faster than the speed of sound. The line tangent to the circles of the speed of sound, known as a Mach wave, forms the boundary between the zones of silence (outside) and action (inside).
  The governing equations for subsonic flow, transonic flow, and supersonic flow are classified as elliptic, parabolic, and hyperbolic, respectively.

Fluid equations classification

The Navier-Stokes equations are a system of non-linear second-order equations in four independent variables. Consequently the classification scheme does not apply directly to them. Nonetheless, the Navier-Stokes equations do possess many of the properties outlined above and the many of the ideas used in solving second-order equations in two independent variables are applicable to them but care must be exercised.

It is possible for a single flow to be described by equations that are not purely of one type.

An important example occurs in steady transonic flows, that is, steady compressible flows that contain both supersonic and subsonic regions. The supersonic regions are hyperbolic in character while the subsonic regions are elliptic. Consequently, it may be necessary to change the method of approximating the equations as a function of the nature of the local flow. To make matters even worse, the regions cannot be determined prior to solving the equations.

Physical boundary conditions

- The boundary conditions, and sometimes the initial conditions, dictate the particular solutions to be obtained from the governing equations.
- For a viscous flow, the boundary condition on a surface assumes zero relative velocity between the surface and the gas immediately at the surface. This is called the no-slip condition.

$$u = v = w = 0$$

- In addition, there is an analogous ‘no-slip’ condition associated with the temperature at the surface.

$$T = T_w$$

T_w is the temperature at the wall.

Physical boundary conditions

- If the wall temperature is not known, e.g., if it is changing as a function of time due to aerodynamic heat transfer to or from the surface, then the Fourier law of heat conduction provides the boundary condition at the surface,

$$\dot{q}_w = -\left(k\frac{\partial T}{\partial n}\right)_w$$

  where n is the direction normal to the wall.

- When the wall temperature becomes such that there is no heat transfer to the surface, this wall temperature, by definition, is called the adiabatic wall temperature. The proper boundary condition for the adiabatic wall is,

$$\left(k\frac{\partial T}{\partial n}\right)_w = 0$$

Physical boundary conditions

- Finally, the only physical boundary conditions along a wall for a continuum viscous flow are the no-slip conditions; these boundary conditions are associated with velocity and temperature at the wall. Other flow properties, such as pressure and density at the wall, fall out as part of the solution.
- For an inviscid flow, there is no friction, and the flow velocity vector immediately adjacent to the wall must be tangent to the wall.

$$\vec{V}\vec{n} = 0$$

  the flow at the surface is tangent to the wall.
- This is the only surface boundary condition for an inviscid flow. The magnitude of the velocity, as well as values of the fluid temperature, pressure, and density at the wall, falls out as part of the solution.

Physical boundary conditions

- Depending on the problem at hand, whether it be viscous or inviscid, there are various types of boundary conditions elsewhere in the flow, away from the surface boundary.
- For example, for flow through a duct of fixed shape, there are boundary conditions which pertain to the inflow and outflow boundaries, such as at the inlet and exit of the duct.
- If the problem involves an aerodynamic body immersed in a known freestream, then the boundary conditions applied at a distance infinitely far upstream, above, below, and downstream of the body are simply that of the given freestream conditions.