2D models

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Outline

Finite Difference formulas

For two-dimensions, we consider

$$\begin{aligned} x_i &= x_0 + i\Delta x \ , \\ y_j &= x_0 + j\Delta y \ , \end{aligned}$$

The forward and backward operators are now given by $\delta_x^{\pm}$ and $\delta_y^{\pm}$ in x and y-directions, respectively. The forward first partial derivatives are

$$\begin{aligned} \left(\frac{\partial u}{\partial x}\right)_{ij} &= \frac{1}{\Delta x}\delta_x^{+} u_{ij} = \frac{u_{i+1,j} - u_{i,j}}{\Delta x} + O(\Delta x) \ , \\ \left(\frac{\partial u}{\partial y}\right)_{ij} &= \frac{1}{\Delta y}\delta_y^{+} u_{ij} = \frac{u_{i,j+1} - u_{i,j}}{\Delta y} + O(\Delta y) \ . \end{aligned}$$

Finite Difference formulas

The second order central difference formulas for the second order derivatives are of the form

$$\left(\frac{\partial^2 u}{\partial x^2}\right)_{ij} = \frac{u_{i+1,j} - 2u_{i,j} + u_{i-1,j}}{\Delta x^2} + O(\Delta x^2) \ ,$$

$$\left(\frac{\partial^2 u}{\partial y^2}\right)_{ij} = \frac{u_{i,j+1} - 2u_{i,j} + u_{i,j-1}}{\Delta y^2} + O(\Delta y^2) \ .$$

An approximation for the mixed derivatives is given by

$$\left(\frac{\partial^2 u}{\partial x \partial y}\right)_{ij} = \frac{u_{i+1,j+1} - u_{i+1,j-1} - u_{i-1,j+1} + u_{i-1,j-1}}{4\Delta x \Delta y} + O\left(\Delta x^2, \Delta y^2\right).$$

Finite Difference formulas

Other approximations are

$$\left(\frac{\partial^2 u}{\partial x \partial y}\right)_{ij} = \frac{u_{i+1,j+1} - u_{i-1,j+1} - u_{i+1,j+1} - u_{i-1,j}}{2\Delta x \Delta y} + O\left(\Delta x^2, \Delta y\right),$$

$$\left(\frac{\partial^2 u}{\partial x \partial y}\right)_{ij} = \frac{u_{i+1,j+1} - u_{i+1,j} - u_{i,j+1} + u_{i,j}}{\Delta x \Delta y} + O\left(\Delta x, \Delta y\right),$$

$$\left(\frac{\partial^2 u}{\partial x \partial y}\right)_{ij} = \frac{u_{i+1,j+1} - u_{i+1,j} - u_{i,j+1} + u_{i-1,j-1} - u_{i-1,j} - u_{i,j-1} + 2u_{i,j}}{\Delta x \Delta y}$$

$$+O\left(\Delta x^2, \Delta y^2\right) \ .$$

Elliptic equations

Let us consider the Poisson equation,

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = -f(x, y) \ ,$$

If we want to solve the equations in a finite domain, Ω, it is necessary to have boundary conditions:

1. $u(\vec{x}) = f(\vec{x})$, $\vec{x} \in \Sigma$, being Σ the boundary of Ω. These are Dirichlet conditions.
2. $\vec{n}\vec{\nabla}u = g(\vec{x})$, being $\vec{n}$ a unitary vector normal to the surface Σ, limiting Ω. These are Neumann bundary conditions.
3. $\vec{n}\vec{\nabla}u + \alpha u = h(\vec{x})$, $\vec{x} \in \Sigma$. These are mixed boundary conditions.

Elliptic equations

The momentum equation for the velocity field $\vec{v}$ in a fluid is

$$\frac{\partial \vec{v}}{\partial t} + (\vec{v} \cdot \nabla)\vec{v} = -\frac{1}{\rho}\nabla p + \nu \nabla^2 \vec{v}$$

Conservation of mass for an incompressible fluid requires that the divergence of $\vec{v}$ must be zero,

$$\vec{\nabla}\vec{v} = 0$$

The momentum equation in x and y components

$$\begin{aligned} \frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} &= -\frac{1}{\rho}\frac{\partial p}{\partial y} + \nu\left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2}\right) \\ \frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} &= -\frac{1}{\rho}\frac{\partial p}{\partial y} + \nu\left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2}\right) \end{aligned}$$

Elliptic equations

Taking the divergence of the momentum equation and applying the incompressibility constraint,

$$\frac{\partial^2 p}{\partial x^2} + \frac{\partial^2 p}{\partial y^2} = -\rho\left(\frac{\partial u}{\partial x}\frac{\partial u}{\partial x} + 2\frac{\partial u}{\partial y}\frac{\partial v}{\partial x} + \frac{\partial v}{\partial y}\frac{\partial v}{\partial y}\right)$$

Which is an equation of the form

$$\boxed{\frac{\partial^2 p}{\partial x^2} + \frac{\partial^2 p}{\partial y^2} = b}$$

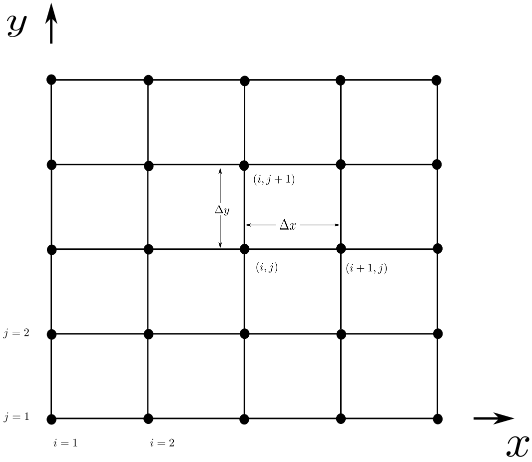
Elliptic equations

Let us consider the following problem

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = -f(x,y) \ , \quad (x,y) \in [0,l_1] \times [0,l_2] \ ,$$
$$u(x,y) = 0 \ , \quad \text{for} \quad x = 0; \ x = l_1; \ y = 0; \ y = l_2 \ .$$

The first step is to consider a mesh in the rectangle $[0,l_1] \times [0,l_2]$.

Elliptic equations



Mesh for a rectangle.

Elliptic equations

We have that

$$\frac{\partial^2 u}{\partial x^2}(u_i, u_j) \approx \frac{u_{i-1j} - 2u_{ij} + u_{i+1j}}{\Delta x^2} \ ,$$
$$\frac{\partial^2 u}{\partial y^2}(u_i, u_j) \approx \frac{u_{ij-1} - 2u_{ij} + u_{ij+1}}{\Delta y^2} \ ,$$

where $u_{ij} = u\left(x_i, y_j\right)$, and the equation

$$\frac{1}{\Delta x^2}\left(u_{i-1j} - 2u_{ij} + u_{i+1j}\right) + \frac{1}{\Delta y^2}\left(u_{ij-1} - 2u_{ij} + u_{ij+1}\right) = -f_{ij} \ .$$

Elliptic equations

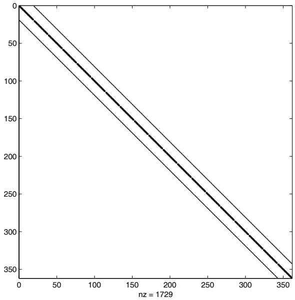
We stablish an order to follow the different nodes of the mesh $i = 1, \ldots, N$, $j = 1, \ldots, M$, for example,

$$l = i + N(j-1) \ .$$

and we obtain a system of linear equations

$$Au = b \ .$$

The dispersity pattern of the matrix A is



Parabolic equations

Let us study the time dependent two-dimensional diffusion equation,

$$\frac{\partial u}{\partial t} = \alpha \left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \right) \ .$$

Using the (FTCS) method, we write an explicit scheme in the form,

$$\frac{u_{i,j}^{n+1} - u_{i,j}^{n}}{\Delta t} = \alpha \left(\frac{u_{i-1,j}^{n} - 2u_{i,j}^{n} + u_{i+1,j}^{n}}{\Delta x^2} + \frac{u_{i,j-1}^{n} - 2u_{i,j}^{n} + u_{i,j+1}^{n}}{\Delta y^2} \right) \ .$$

It can be shown that the system is stable if

$$d_x + d_y \leq \frac{1}{2} \ ,$$

where

$$d_x = \frac{\alpha \Delta t}{\Delta x^2}, \quad d_y = \frac{\alpha \Delta t}{\Delta y^2}.$$

Parabolic equations

To avoid the stability restrictions, we can use an implicit scheme

$$\frac{u_{i,j}^{n+1} - u_{i,j}^{n}}{\Delta t} = \alpha \left(\frac{u_{i-1,j}^{n+1} - 2u_{i,j}^{n+1} + u_{i+1,j}^{n+1}}{\Delta x^2} + \frac{u_{i,j-1}^{n+1} - 2u_{i,j}^{n+1} + u_{i,j+1}^{n+1}}{\Delta y^2} \right) ,$$

or

$$d_x u_{i+1,j}^{n+1} + d_x u_{i-1,j}^{n+1} - (2d_x + 2d_y + 1) u_{i,j}^{n+1} + d_y u_{i,j+1}^{n+1} + d_y u_{i,j-1}^{n+1} = -u_{i,j}^{n} ,$$

which, after an adequate ordering of the nodes leads to a pentadiagonal system, which should be solved for each time step.

Parabolic equations

An alternative is to use the alternating direction implicit (ADI) scheme,

$$\frac{u_{i,j}^{n+\frac{1}{2}} - u_{i,j}^{n}}{\Delta t/2} = \alpha \left(\frac{u_{i-1,j}^{n+\frac{1}{2}} - 2u_{i,j}^{n+\frac{1}{2}} + u_{i+1,j}^{n+\frac{1}{2}}}{\Delta x^2} + \frac{u_{i-1,j}^{n} - 2u_{i,j}^{n} + u_{i+1,j}^{n}}{\Delta y^2} \right)$$

$$\frac{u_{i,j}^{n+1} - u_{i,j}^{n+\frac{1}{2}}}{\Delta t/2} = \alpha \left(\frac{u_{i-1,j}^{n+\frac{1}{2}} - 2u_{i,j}^{n+\frac{1}{2}} + u_{i+1,j}^{n+\frac{1}{2}}}{\Delta x^2} + \frac{u_{i-1,j}^{n+1} - 2u_{i,j}^{n+1} + u_{i+1,j}^{n+1}}{\Delta y^2} \right)$$

Parabolic equations

This scheme is unconditionally stable and can be written in tridiagonal form

$$-d_1 u_{i-1,j}^{n+\frac{1}{2}} + (1+2d_1)u_{i,j}^{n+\frac{1}{2}} - d_1 u_{i+1,j}^{n+\frac{1}{2}} = d_2 u_{i,j-1}^{n} + (1-2d_2)u_{i,j}^{n} + d_2 u_{i,j-1}^{n}$$

$$-d_2 u_{i,j-1}^{n+1} + (1+2d_2)u_{i,j}^{n+1} - d_2 u_{i,j+1}^{n+1} = d_1 u_{i+1,j}^{n+\frac{1}{2}} + (1-2d_1)u_{i,j}^{n+\frac{1}{2}} + d_1 u_{i+1,j}^{n+\frac{1}{2}} ,$$

where

$$d_1 = \frac{\alpha \Delta t}{2\Delta x^2} , \quad d_2 = \frac{\alpha \Delta t}{2\Delta y^2} .$$

Both systems can be written as tridiagonal systems if a different order is used for numbering the nodes in the mesh (the role of the rows and columns is swapped).

The Crank-Nicolson scheme can be written in two steps,

$$\frac{u_{i,j}^{n+\frac{1}{2}} - u_{i,j}^{n}}{\Delta t/2} = \frac{\alpha}{2}\left(\frac{u_{i-1,j}^{n+\frac{1}{2}} - 2u_{i,j}^{n+\frac{1}{2}} + u_{i+1,j}^{n+\frac{1}{2}}}{\Delta x^2} + \frac{u_{i-1,j}^{n} - 2u_{i,j}^{n} + u_{i+1,j}^{n}}{\Delta x^2}\right)$$

$$\frac{u_{i,j}^{n+1} - u_{i,j}^{n+\frac{1}{2}}}{\Delta t/2} = \frac{\alpha}{2}\left(\frac{u_{i,j-1}^{n+1} - 2u_{i,j}^{n+1} + u_{i,j+1}^{n+1}}{\Delta y^2} + \frac{u_{i,j-1}^{n+\frac{1}{2}} - 2u_{i,j}^{n+\frac{1}{2}} + u_{i,j+1}^{n+\frac{1}{2}}}{\Delta y^2}\right)$$

which is an unconditionally stable scheme.

Parabolic equations

Given the two-dimensional diffusion equation

$$\frac{\partial u}{\partial t} = \alpha_x \frac{\partial^2 u}{\partial x^2} + \alpha_y \frac{\partial^2 u}{\partial y^2} \ ,$$

a general two-level implicit finite differences scheme is

$$\begin{aligned}\frac{\Delta u^{n+1}}{\Delta t} \quad &= \quad (1-\beta)\left(\alpha_x L_{xx} u_{i,j}^n + \alpha_y L_{yy} u_{i,j}^n\right) \\ &\quad +\beta\left(\alpha_x L_{xx} u_{i,j}^{n+1} + \alpha_y L_{yy} u_{i,j}^{n+1}\right) \ ,\end{aligned}$$

where $L_{xx} u_{i,j}^n = \frac{u_{i-1,j}^n - 2u_{i,j}^n + u_{i+1,j}^n}{\Delta x^2}$, $L_{yy} u_{i,j}^n = \frac{u_{i,j-1}^n - 2u_{i,j}^n + u_{i,j+1}^n}{\Delta y^2}$

Parabolic equations

Making use of the Taylor expansion

$$u_{i,j}^{n+1} = u_{i,j}^{n} + \Delta t \left(\frac{\partial u}{\partial t} \right)_{i,j}^{n} + O\left(\Delta t^2\right) \;,$$

which is approximated by

$$u_{i,j}^{n+1} = u_{i,j}^{n} + \Delta t \left(\frac{\Delta u}{\Delta t} \right)_{i,j}^{n} + O\left(\Delta t^2\right) \;,$$

substituting this result

$$\begin{aligned} \frac{\Delta u^{n+1}}{\Delta t} &= \left(\alpha_x L_{xx} u_{i,j}^{n} + \alpha_y L_{yy} u_{i,j}^{n}\right) \\ &\quad + \beta \left(\alpha_x L_{xx} \Delta u_{i,j}^{n+1} + \alpha_y L_{yy} \Delta u_{i,j}^{n+1}\right) \;, \end{aligned}$$

Parabolic equations

After rearrangement,

$$\left(1-\beta\Delta t\left(\alpha_{xx}L_{xx}+\alpha_y L_{yy}\right)\right)\Delta u_{i,j}^{n+1}=\Delta t\left(\alpha_{xx}L_{xx}+\alpha_y L_{yy}\right)u_{i,j}^{n}$$

Algebraic operators appropriate to both directions appear.

In order to be able to solve tridiagonal systems it is replaced by the approximate factorisation

$$\left(1-\beta\Delta t\alpha_x L_{xx}\right)\left(1-\beta\Delta t\alpha_y L_{yy}\right)\Delta u_{i,j}^{n+1}=\Delta t\left(\alpha_x L_{xx}+\alpha_y L_{yy}\right)u_{i,j}^{n}\ .$$

In this factorisation an extra term appears

$$\beta^2\Delta t^2\alpha_x\alpha_y L_{xx}L_{yy}\Delta u_{i,j}^{n+1}=O\left(\Delta t^2\right)\ .$$

Parabolic equations

This equation is solved in two steps:

$$
\begin{aligned}
(1 - \beta \Delta t \alpha_x L_{xx})\, \Delta u^*_{i,j} &= \Delta t\, (\alpha_x L_{xx} + \alpha_y L_{yy})\, u^n_{i,j} \ , \\
(1 - \beta \Delta t \alpha_y L_{yy})\, \Delta u^{n+1}_{i,j} &= \Delta u^*_{i,j} \ .
\end{aligned}
$$

Advection-diffusion equation

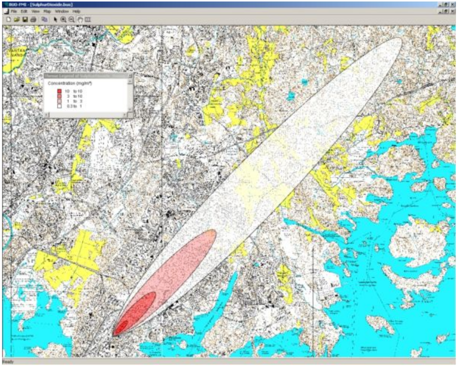
Sometimes, we have a fluid which diffusion takes place and it is also moving in a preferential direction. The obvious cases are those of a flowing river and of a smokestack plume being blown by the wind.

For a 2D problem we have that for the concentration of a substance, C, satisfies de Advection-diffusion equation

$$\frac{\partial C}{\partial t} + u\frac{\partial C}{\partial x} + v\frac{\partial C}{\partial y} = D\left(\frac{\partial^2 C}{\partial x^2} + \frac{\partial^2 C}{\partial y^2}\right)$$

Advection-diffusion equation

Simulation of the effect of a point-source in Finland
(by the Finnish Meteorological Institute)



Advection-diffusion equation

Example

Given the problem

$$\frac{\partial u}{\partial t} = 0.5\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right) - 5\left(\frac{\partial u}{\partial x} + \frac{\partial u}{\partial xy}\right)$$

with homogeneous boundary conditions in $[0,1] \times [0,1]$ and initial condition

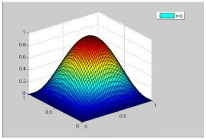
$$u(x,y,0) = \sin(\pi x)\sin(\pi y)$$

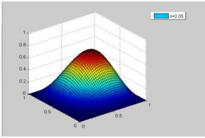
Using the variables separation method, the analytical solution for this problem is

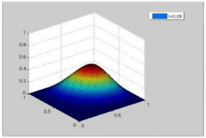
$$u(x,y,t) = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} A_{m,n} e^{\left(-0.5\pi^2(m^2+n^2)-25\right)t} e^{5(x+y)} \sin(m\pi x)\sin(n\pi x)$$

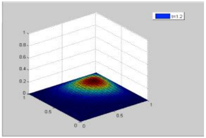
With

$$A_{m,n} = 4\int_0^1\int_0^1 e^{-5(x+y)}\sin(\pi x)\sin(\pi y)\sin(m\pi x)\sin(n\pi y)\,dxdy$$









Advection-diffusion equation

Let us consider the generic problem associated with the advection diffusion equation,

$$\frac{\partial u}{\partial t} + \beta_1 \frac{\partial u}{\partial x} + \beta_2 \frac{\partial u}{\partial y} = \alpha_1 \frac{\partial^2 u}{\partial x^2} + \alpha_2 \frac{\partial^2 u}{\partial y^2} \ , \quad 0 \le x \le 1, \quad 0 \le y \le 1, \quad t \ge 0,$$

with the initial condition, $u(x, y, 0) = f(x, y)$, and the boundary conditions

$$\begin{aligned} u(0, y, t) = g_1(y, t) \ , \quad u(1, y, t) = g_2(y, t) \ , \\ u(x, 0, t) = h_1(x, t) \ , \quad u(x, 1, t) = h_2(y, t) \ , \end{aligned}$$

An ADI method for this problem is given by,

$$\frac{u_{i,j}^{*}-u_{i,j}^{n}}{\Delta t/2}+\beta_1\frac{u_{i+1,j}^{*}-u_{i-1,j}^{*}}{2\Delta x}+\beta_2\frac{u_{i,j+1}^{n}-u_{i,j-1}^{*}}{2\Delta y}=$$
$$\alpha_1\frac{u_{i+1,j}^{*}-2u_{i,j}^{*}+u_{i-1,j}^{*}}{\Delta x^2}+\alpha_2\frac{u_{i,j-1}^{n}-2u_{i,j}^{n}+u_{i,j+11}^{*}}{\Delta y^2}$$

and

$$\frac{u_{i,j}^{n+1}-u_{i,j}^{*}}{\Delta t/2}+\beta_1\frac{u_{i+1,j}^{*}-u_{i-1,j}^{*}}{2\Delta x}+\beta_2\frac{u_{i,j+1}^{n+1}-u_{i,j-1}^{n+1}}{2\Delta y}=$$
$$\alpha_1\frac{u_{i-1,j}^{*}-2u_{i,j}^{*}+u_{i+1,j}^{*}}{\Delta x^2}+\alpha_2\frac{u_{i,j-1}^{n+1}-2u_{i,j}^{n+1}+u_{i,j+11}^{n+1}}{\Delta y^2}\ .$$

A Crank-Nicolson scheme is

$$\frac{u_{i,j}^{n+1}-u_{i,j}^{n}}{\Delta t}=\frac{\beta_1}{2}\left(\frac{u_{i+1,j}^{n+1}-u_{i-1,j}^{n+1}}{2\Delta x}+\frac{u_{i+1,j}^{n}-u_{i-1,j}^{n}}{2\Delta x}\right)+$$

$$\frac{\beta_2}{2}\left(\frac{u_{i,j+1}^{n+1}-u_{i,j-1}^{n+1}}{2\Delta y}+\frac{u_{i,j+1}^{n}-u_{i,j-1}^{n}}{2\Delta y}\right)=$$

$$\frac{\alpha_1}{2}\left(\frac{u_{i-1,j}^{n+1}-2u_{i,j}^{n+1}+u_{i+1,j}^{n+1}}{\Delta x^2}+\frac{u_{i-1,j}^{n}-2u_{i,j}^{n}+u_{i+1,j}^{n}}{\Delta x^2}\right)$$

$$+\frac{\alpha_2}{2}\left(\frac{u_{i,j-1}^{n+1}-2u_{i,j}^{n+1}+u_{i,j+1}^{n+1}}{\Delta y^2}+\frac{u_{i,j-1}^{n}-2u_{i,j}^{n}+u_{i,j+1}^{n}}{\Delta y^2}\right)\ .$$

Advection-diffusion equation

Introducing the operators

$$L_{xx}u_{i,j} = u_{i-1,j} - 2u_{i,j} + u_{i+1,j} \ , \quad L_{yy}u_{i,j} = u_{i,j-1} - 2u_{i,j} + u_{i,j+1} \ ,$$
$$L_{x}u_{i,j} = u_{i+1,j} - u_{i-1,j} \ , \quad L_{y}u_{i,j} = u_{i,j+1} - u_{i,j-1} \ ,$$

and

$$\mu_i = \frac{\alpha_i \Delta t}{2\Delta x^2} \ , \quad \sigma_i = \frac{\beta_i \Delta t}{4\Delta x} \ ,$$

the scheme can be written as

$$\left(1 - \mu_x L_{xx} - \mu_y L_{yy} + \sigma_x L_x + \sigma_y L_y\right) u_{i,j}^{n+1} =$$
$$\left(1 - \mu_x L_{xx} - \mu_y L_{yy} + \sigma_x L_x + \sigma_y L_y\right) u_{i,j}^{n} \ ,$$

which is solved using the factorization

$$\left(1 - \mu_x L_{xx} + \sigma_x L_x\right)\left(1 - \mu_y L_{yy} + \sigma_y L_y\right) u_{i,j}^{n+1} =$$
$$\left(1 - \mu_x L_{xx} + \sigma_x L_x\right)\left(1 - \mu_y L_{yy} + \sigma_y L_y\right) u_{i,j}^{n} \ .$$

Coordinate Transformation for Arbitrary Geometries

Let us consider a two-dimensional coordinate system of the physical domain (x, y), and the computational domain (ξ, η). We begin with spatial derivatives of any variable with respect ξ and η,

$$
\begin{aligned}
\frac{\partial}{\partial \xi} &= \frac{\partial}{\partial x}\frac{\partial x}{\partial \xi} + \frac{\partial}{\partial y}\frac{\partial y}{\partial \xi} \,, \\
\frac{\partial}{\partial \eta} &= \frac{\partial}{\partial x}\frac{\partial x}{\partial \eta} + \frac{\partial}{\partial y}\frac{\partial y}{\partial \eta} \,.
\end{aligned}
$$

In matrix form

$$
\begin{pmatrix} \frac{\partial}{\partial \xi} \\ \frac{\partial}{\partial \eta} \end{pmatrix} = J \begin{pmatrix} \frac{\partial}{\partial x} \\ \frac{\partial}{\partial y} \end{pmatrix} \,,
$$

where the Jacobian matrix (transpose) is

$$J = \begin{pmatrix} \frac{\partial x}{\partial \xi} & \frac{\partial x}{\partial \xi} \\ \frac{\partial x}{\partial \eta} & \frac{\partial x}{\partial \eta} \end{pmatrix} .$$

Thus,

$$\begin{pmatrix} \frac{\partial}{\partial x} \\ \frac{\partial}{\partial y} \end{pmatrix} = \frac{1}{|J|} \begin{pmatrix} \frac{\partial y}{\partial \eta} & -\frac{\partial y}{\partial \xi} \\ -\frac{\partial x}{\partial \eta} & \frac{\partial x}{\partial \xi} \end{pmatrix} \begin{pmatrix} \frac{\partial}{\partial \xi} \\ \frac{\partial}{\partial \eta} \end{pmatrix} .$$

Coordinate Transformation for Arbitrary Geometries

The second derivatives,

$$
\begin{aligned}
\frac{\partial^2}{\partial x^2} \quad = \quad & \frac{1}{|J|^2}\left(\left(\frac{\partial y}{\partial \eta}\right)^2 \frac{\partial^2}{\partial \xi^2} - 2\frac{\partial y}{\partial \eta}\frac{\partial y}{\partial \xi}\frac{\partial^2}{\partial \xi \partial \eta} + \left(\frac{\partial y}{\partial \xi}\right)^2 \frac{\partial^2}{\partial \eta^2}\right. \\
& \left.\left(\frac{\partial y}{\partial \eta}\frac{\partial^2 y}{\partial \xi \partial \eta} - \frac{\partial y}{\partial \xi}\frac{\partial^2 y}{\partial \eta^2}\right)\frac{\partial}{\partial \xi} + \left(\frac{\partial y}{\partial \xi}\frac{\partial^2 y}{\partial \xi \partial \eta} - \frac{\partial y}{\partial \eta}\frac{\partial^2 y}{\partial \xi^2}\right)\frac{\partial}{\partial \eta}\right) \\
& - \frac{1}{|J|^3}\left(\left(\frac{\partial y}{\partial \eta}\right)^2 \frac{\partial |J|}{\partial \xi}\frac{\partial}{\partial \xi} - \frac{\partial y}{\partial \eta}\frac{\partial y}{\partial \xi}\frac{\partial |J|}{\partial \xi}\frac{\partial}{\partial \eta}\right. \\
& \left. - \frac{\partial y}{\partial \xi}\frac{\partial y}{\partial \eta}\frac{\partial |J|}{\partial \eta}\frac{\partial}{\partial \xi} + \left(\frac{\partial y}{\partial \xi}\right)^2 \frac{\partial |J|}{\partial \eta}\frac{\partial}{\partial \eta}\right) \quad ,
\end{aligned}
$$

Coordinate Transformation for Arbitrary Geometries

$$
\begin{aligned}
\frac{\partial^2}{\partial y^2} \;=\;& \frac{1}{|J|^2}\left(\left(\frac{\partial x}{\partial \eta}\right)^2 \frac{\partial^2}{\partial \xi^2} - 2\frac{\partial x}{\partial \eta}\frac{\partial x}{\partial \xi}\frac{\partial^2}{\partial \xi \partial \eta} + \left(\frac{\partial x}{\partial \xi}\right)^2 \frac{\partial^2}{\partial \eta^2}\right. \\
& \left.\left(\frac{\partial x}{\partial \eta}\frac{\partial^2 x}{\partial \xi \partial \eta} - \frac{\partial x}{\partial \xi}\frac{\partial^2 x}{\partial \eta^2}\right)\frac{\partial}{\partial \xi} + \left(\frac{\partial x}{\partial \xi}\frac{\partial^2 x}{\partial \xi \partial \eta} - \frac{\partial x}{\partial \eta}\frac{\partial^2 x}{\partial \xi^2}\right)\frac{\partial}{\partial \eta}\right) \\
& - \frac{1}{|J|^3}\left(\left(\frac{\partial x}{\partial \eta}\right)^2 \frac{\partial |J|}{\partial \xi}\frac{\partial}{\partial \xi} - \frac{\partial x}{\partial \eta}\frac{\partial x}{\partial \xi}\frac{\partial |J|}{\partial \xi}\frac{\partial}{\partial \eta}\right. \\
& \left. - \frac{\partial x}{\partial \xi}\frac{\partial x}{\partial \eta}\frac{\partial |J|}{\partial \eta}\frac{\partial}{\partial \xi} + \left(\frac{\partial x}{\partial \xi}\right)^2 \frac{\partial |J|}{\partial \eta}\frac{\partial}{\partial \eta}\right) \;,
\end{aligned}
$$

Coordinate Transformation for Arbitrary Geometries

where

$$
\begin{aligned}
\frac{\partial |J|}{\partial \xi} &= \frac{\partial}{\partial \xi}\left(\frac{\partial x}{\partial \xi}\frac{\partial y}{\partial \xi} - \frac{\partial y}{\partial \xi}\frac{\partial x}{\partial \eta}\right) \\
&= \frac{\partial^2 x}{\partial \xi^2}\frac{\partial y}{\partial \eta} + \frac{\partial x}{\partial \xi}\frac{\partial^2 y}{\partial \xi \partial \eta} - \frac{\partial^2 y}{\partial \xi^2}\frac{\partial x}{\partial \eta} - \frac{\partial y}{\partial \xi}\frac{\partial^2 x}{\partial \xi \partial \eta}
\end{aligned}
$$

Coordinate Transformation for Arbitrary Geometries

Let us consider the vector convection-diffusion equation

$$\frac{\partial U}{\partial t} + u\frac{\partial U}{\partial x} + v\frac{\partial U}{\partial y} - \nu\left(\frac{\partial^2 U}{\partial x^2} + \frac{\partial^2 U}{\partial y^2}\right) = F \ ,$$

with

$$U = \begin{pmatrix} u \\ v \end{pmatrix} \ , \quad F = \begin{pmatrix} f_x \\ f_y \end{pmatrix} \ .$$

Coordinate Transformation for Arbitrary Geometries

Making the change from variables (x, y) to (ξ, η),

$$\frac{\partial U}{\partial t} + \bar{u}\frac{\partial U}{\partial \xi} + \bar{v}\frac{\partial U}{\partial y} - \nu\left(\frac{1}{|J|^2}\left(a\frac{\partial^2 U}{\partial \xi^2} - 2b\frac{\partial^2 U}{\partial \xi \partial \eta} + c\frac{\partial^2 U}{\partial \eta^2}\right)\right.$$
$$\left. + p\frac{\partial U}{\partial \xi} + q\frac{\partial U}{\partial \eta}\right) = F\ ,$$

where

$$\bar{u} = \frac{1}{|J|}\left(u\frac{\partial y}{\partial \eta} - v\frac{\partial x}{\partial \eta}\right)\ ,$$
$$\bar{v} = \frac{1}{|J|}\left(v\frac{\partial x}{\partial \xi} - u\frac{\partial y}{\partial \xi}\right)\ ,$$
$$p = \frac{1}{|J|^3}\left(-\frac{\partial y}{\partial \eta}\left(a\frac{\partial^2 x}{\partial \xi^2} - 2b\frac{\partial^2 x}{\partial \xi \partial \eta} + c\frac{\partial^2 x}{\partial \eta^2}\right)\right.$$
$$\left. + \frac{\partial x}{\partial \eta}\left(a\frac{\partial^2 y}{\partial \xi^2} - 2b\frac{\partial^2 y}{\partial \xi \partial \eta} + c\frac{\partial^2 y}{\partial \eta^2}\right)\right)$$

Coordinate Transformation for Arbitrary Geometries

$$
\begin{aligned}
q &= \frac{1}{|J|^3}\left(\frac{\partial y}{\partial \xi}\left(a\frac{\partial^2 x}{\partial \xi^2} - 2b\frac{\partial^2 x}{\partial \xi \partial \eta} + c\frac{\partial^2 x}{\partial \eta^2}\right)\right. \\
&\quad \left. -\frac{\partial x}{\partial \xi}\left(a\frac{\partial^2 y}{\partial \xi^2} - 2b\frac{\partial^2 y}{\partial \xi \partial \eta} + c\frac{\partial^2 y}{\partial \eta^2}\right)\right) \\
a &= \left(\frac{\partial x}{\partial \eta}\right)^2 + \left(\frac{\partial y}{\partial \eta}\right)^2 \ , \\
b &= \frac{\partial x}{\partial \xi}\frac{\partial x}{\partial \eta} + \frac{\partial y}{\partial \xi}\frac{\partial y}{\partial \eta} \ , \\
c &= \left(\frac{\partial x}{\partial \xi}\right)^2 + \left(\frac{\partial y}{\partial \xi}\right)^2 \ .
\end{aligned}
$$

Coordinate Transformation for Arbitrary Geometries

These equations may be solved using the predictor-corrector MacCormack method,

Predictor

$$
\begin{aligned}
U^*_{i,j} &= U^n_{i,j} + \Delta t \left(- \left(\bar{u}\frac{\partial U}{\partial \xi} + \bar{v}\frac{\partial U}{\partial \eta} \right)^n_{i,j} \right. \\
&\quad v\Delta t \left(\frac{1}{|J|^2} \left(a\frac{\partial^2 U}{\partial \xi^2} - 2b\frac{\partial^2 U}{\partial \xi \partial \eta} + c\frac{\partial^2 U}{\partial \eta^2} \right) + p\frac{\partial U}{\partial \xi} + q\frac{\partial U}{\partial \xi} \right)^n_{i,j} \\
&\quad \left. + F^n_{i,j} \right)
\end{aligned}
$$

Corrector

$$U_{i,j}^{n+1} = \frac{1}{2}\left(U_{i,j}^{*} + U_{i,j}^{n}\right) + \frac{\Delta t}{2}\left(-\left(\bar{u}\frac{\partial U}{\partial \xi} + \bar{v}\frac{\partial U}{\partial \eta}\right)_{i,j}^{*}\right)$$
$$+\frac{\nu \Delta t}{2}\left(\frac{1}{|J|^2}\left(a\frac{\partial^2 U}{\partial \xi^2} - 2b\frac{\partial^2 U}{\partial \xi \partial \eta} + c\frac{\partial^2 U}{\partial \eta^2}\right) + p\frac{\partial U}{\partial \xi} + q\frac{\partial U}{\partial \eta}\right)_{i,j}^{*}$$
$$+\frac{\Delta t}{2}F_{i,j}^{n+1} \ .$$

Coordinate Transformation for Arbitrary Geometries

Example

Write the advection-diffusion equation

$$\frac{\partial u}{\partial t} + \beta_1 \frac{\partial u}{\partial x} + \beta_2 \frac{\partial u}{\partial y} = \alpha \left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \right) ,$$

using the polar coordinates:

$$x = r\cos(\theta), \quad y = r\sin(\theta) .$$

Solution:

Using the Chain's Rule

$$\begin{aligned}
\frac{\partial u}{\partial x} &= \frac{\partial u}{\partial r}\frac{\partial r}{\partial x} + \frac{\partial u}{\partial \theta}\frac{\partial \theta}{\partial x} \\
\frac{\partial u}{\partial y} &= \frac{\partial u}{\partial r}\frac{\partial r}{\partial y} + \frac{\partial u}{\partial \theta}\frac{\partial \theta}{\partial y}
\end{aligned}$$

Coordinate Transformation for Arbitrary Geometries

We start from

$$\begin{aligned} x &= r\cos(\theta) \\ y &= r\sin(\theta) \end{aligned}$$

Derivating with respect to x

$$\begin{aligned} 1 &= \frac{\partial r}{\partial x}\cos(\theta) - r\sin(\theta)\frac{\partial \theta}{\partial x} \\ 0 &= \frac{\partial r}{\partial x}\sin(\theta) + r\cos(\theta)\frac{\partial \theta}{\partial x} \end{aligned}$$

Solving the system

$$\boxed{\frac{\partial r}{\partial x} = \cos(\theta)\ , \quad \frac{\partial \theta}{\partial x} = -\frac{1}{r}\sin(\theta)}$$

Derivating with respect to y

$$
\begin{aligned}
0 &= \frac{\partial r}{\partial y}\cos(\theta) - r\sin(\theta)\frac{\partial \theta}{\partial y} \\
1 &= \frac{\partial r}{\partial y}\sin(\theta) + r\cos(\theta)\frac{\partial \theta}{\partial y}
\end{aligned}
$$

Solving the system

$$
\boxed{\frac{\partial r}{\partial y} = \sin(\theta)\ , \quad \frac{\partial \theta}{\partial x} = \frac{1}{r}\cos(\theta)}
$$

Coordinate Transformation for Arbitrary Geometries

We obtain

$$
\begin{aligned}
\frac{\partial u}{\partial x} &= \cos(\theta)\frac{\partial u}{\partial r} - \frac{1}{r}\sin(\theta)\frac{\partial u}{\partial \theta} \\
\frac{\partial u}{\partial y} &= \sin(\theta)\frac{\partial u}{\partial r} + \frac{1}{r}\cos(\theta)\frac{\partial u}{\partial \theta}
\end{aligned}
$$

Coordinate Transformation for Arbitrary Geometries

$$
\begin{aligned}
\frac{\partial^2 u}{\partial x^2} &= \frac{\partial}{\partial r}\left(\cos(\theta)\frac{\partial u}{\partial r} - \frac{1}{r}\sin(\theta)\frac{\partial u}{\partial \theta}\right)\frac{\partial r}{\partial x} \\
&\quad \frac{\partial}{\partial \theta}\left(\cos(\theta)\frac{\partial u}{\partial r} - \frac{1}{r}\sin(\theta)\frac{\partial u}{\partial \theta}\right)\frac{\partial \theta}{\partial x} \\
&= \cos^2(\theta)\frac{\partial^2 u}{\partial r^2} + \frac{2}{r^2}\sin(\theta)\cos(\theta)\frac{\partial u}{\partial \theta} - \frac{2}{r}\sin(\theta)\cos(\theta)\frac{\partial^2 u}{\partial r \partial \theta} \\
&\quad + \frac{1}{r}\sin^2(\theta)\frac{\partial u}{\partial r} + \frac{1}{r^2}\sin^2(\theta)\frac{\partial^2 u}{\partial \theta^2}
\end{aligned}
$$

Coordinate Transformation for Arbitrary Geometries

$$
\begin{aligned}
\frac{\partial^2 u}{\partial y^2} &= \frac{\partial}{\partial r}\left(\sin(\theta)\frac{\partial u}{\partial r} + \frac{1}{r}\cos(\theta)\frac{\partial u}{\partial \theta}\right)\frac{\partial r}{\partial y} \\
&\quad \frac{\partial}{\partial \theta}\left(\sin(\theta)\frac{\partial u}{\partial r} + \frac{1}{r}\cos(\theta)\frac{\partial u}{\partial \theta}\right)\frac{\partial \theta}{\partial y} \\
&= \sin^2(\theta)\frac{\partial^2 u}{\partial r^2} - \frac{2}{r^2}\sin(\theta)\cos(\theta)\frac{\partial u}{\partial \theta} + \frac{2}{r}\sin(\theta)\cos(\theta)\frac{\partial^2 u}{\partial r \partial \theta} \\
&\quad + \frac{1}{r}\cos^2(\theta)\frac{\partial u}{\partial r} + \frac{1}{r^2}\cos^2(\theta)\frac{\partial^2 u}{\partial \theta^2}
\end{aligned}
$$

Coordinate Transformation for Arbitrary Geometries

The equation

$$\frac{\partial u}{\partial t} + \beta_1 \frac{\partial u}{\partial x} + \beta_2 \frac{\partial u}{\partial y} = \alpha \left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \right)$$

$$\frac{\partial u}{\partial t} + \beta_1 \left(\cos(\theta) \frac{\partial u}{\partial r} - \frac{1}{r} \sin(\theta) \frac{\partial u}{\partial \theta} \right) + \beta_2 \left(\sin(\theta) \frac{\partial u}{\partial r} + \frac{1}{r} \cos(\theta) \frac{\partial u}{\partial \theta} \right)$$

$$= \alpha \left(\frac{\partial^2 u}{\partial r^2} + \frac{1}{r} \frac{\partial u}{\partial r} + \frac{1}{r^2} \frac{\partial^2 u}{\partial \theta^2} \right)$$

Fluid equations

We consider an unsteady two-dimensional inviscid flow

$$\begin{aligned}
&\text{Continuity} && \frac{\partial \rho}{\partial t} = -\left(\rho\frac{\partial v_x}{\partial x} + v_x\frac{\partial \rho}{\partial x} + \rho\frac{\partial v_y}{\partial y} + v_y\frac{\partial \rho}{\partial y}\right) \\
&x-\text{momentum} && \frac{\partial v_x}{\partial t} = -\left(v_x\frac{\partial v_x}{\partial x} + v_y\frac{\partial v_y}{\partial y} + \frac{1}{\rho}\frac{\partial p}{\partial x}\right) \\
&y-\text{momentum} && \frac{\partial v_y}{\partial t} = -\left(v_x\frac{\partial v_y}{\partial x} + v_y\frac{\partial v_y}{\partial y} + \frac{1}{\rho}\frac{\partial p}{\partial y}\right) \\
&\text{Energy} && \frac{\partial e}{\partial t} = -\left(v_x\frac{\partial e}{\partial x} + v_y\frac{\partial e}{\partial y} + \frac{p}{\rho}\frac{\partial v_x}{\partial x} + \frac{p}{\rho}\frac{\partial v_y}{\partial y}\right)
\end{aligned}$$

Fluid equations

To obtain an explicit Lax-Wendroff method we use the Taylor expansions

$$
\begin{aligned}
\rho_{i,j}^{n+1} &= \rho_{i,j}^{n} + \left(\frac{\partial \rho}{\partial t}\right)_{i,j}^{n} \Delta t + \left(\frac{\partial^2 \rho}{\partial t^2}\right)_{i,j}^{n} \frac{\Delta t^2}{2} + \cdots \\
(v_x)_{i,j}^{n+1} &= (v_x)_{i,j}^{n} + \left(\frac{\partial v_x}{\partial t}\right)_{i,j}^{n} \Delta t + \left(\frac{\partial^2 v_x}{\partial t^2}\right)_{i,j}^{n} \frac{\Delta t^2}{2} + \cdots \\
&\vdots
\end{aligned}
$$

Fluid equations

For example, using the continuity equation

$$\left(\frac{\partial \rho}{\partial t}\right)^n_{i,j} = -\left(\rho^n_{i,j}\frac{(v_x)^n_{i+1,j}-(v_x)^n_{i-1,j}}{2\Delta x} + (v_x)^n_{i,j}\,\frac{\rho^n_{i+1,j}-\rho^n_{i-1,j}}{2\Delta x}\right.$$
$$\left.\rho^n_{i,j}\frac{(v_y)^n_{i,j+1}-(v_y)^n_{i,j-1}}{2\Delta y} + (v_y)^n_{i,j}\,\frac{\rho^n_{i,j+1}-\rho^n_{i,j-1}}{2\Delta y}\right)$$

The second derivative

$$\frac{\partial^2 \rho}{\partial t^2} = -\left(\rho\frac{\partial^2 v_x}{\partial x\partial t} + \frac{\partial v_x}{\partial x}\frac{\partial \rho}{\partial t} + v_x\frac{\partial^2 \rho}{\partial x\partial t} + \frac{\partial \rho}{\partial x}\frac{\partial v_x}{\partial t}\right.$$
$$\left.+\rho\frac{\partial^2 v_y}{\partial y\partial t} + \frac{\partial v_y}{\partial y}\frac{\partial \rho}{\partial t} + v_y\frac{\partial^2 \rho}{\partial y\partial t} + \frac{\partial \rho}{\partial y}\frac{\partial v_y}{\partial t}\right)$$

Fluid equations

The mixed derivatives

$$\frac{\partial^2 v_x}{\partial x \partial t} = -\left(v_x \frac{\partial^2 v_x}{\partial x^2} + \left(\frac{\partial v_x}{\partial x}\right)^2 + v_y \frac{\partial^2 v_x}{\partial x \partial y} + \frac{\partial v_x}{\partial y}\frac{\partial v_y}{\partial x} + \frac{1}{\rho}\frac{\partial^2 p}{\partial x^2} - \frac{1}{\rho^2}\frac{\partial p}{\partial x}\frac{\partial \rho}{\partial x}\right)$$

Using central differences

$$\left(\frac{\partial^2 v_x}{\partial x \partial t}\right)_{i,j}^n = -\left((v_x)_{i,j}^n \frac{(v_x)_{i-1,j}^n - 2(v_x)_{i,j}^n + (v_x)_{i+1,j}^n}{\Delta x^2} + \left(\frac{(v_x)_{i+1,j}^n - (v_x)_{i-1,j}^n}{2\Delta x}\right)^2 + (v_y)_{i,j}^n \frac{(v_x)_{i+1,j+1}^n + (v_x)_{i-1,j-1}^n - (v_x)_{i-1,j+1}^n - (v_x)_{i+1,j-1}^n}{4\Delta x \Delta y} + \frac{(v_x)_{i,j+1}^n - (v_x)_{i,j-1}^n}{2\Delta y}\frac{(v_y)_{i+1,j}^n - (v_x)_{i-1,j}^n}{2\Delta x} + \frac{1}{\rho_{i,j}^n}\frac{p_{i-1,j}^n - 2p_{i,j}^n + p_{i+1,j}^n}{\Delta x^2} - \frac{1}{\left(\rho_{i,j}^n\right)^2}\frac{p_{i+1,j}^n - p_{i-1,j}^n}{2\Delta x}\frac{\rho_{i+1,j}^n - \rho_{i-1,j}^n}{2\Delta x}\right)$$

Fluid equations

- The same procedure is followed for the other variables, obtaining a second order accurate method in time and in space.
- The need of using the second derivatives in the Taylor expansions makes necessary to use long equations, and this makes this method unpopular.

Fluid equations

For the density

$$\rho_{i,j}^{n+1} = \rho_{i,j}^{n} + \left(\frac{\partial \rho}{\partial t}\right)_{\text{av}} \Delta t$$

$\left(\frac{\partial \rho}{\partial t}\right)_{\text{av}}$ is a representative mean value of $\partial\rho/\partial t$ between t and $t + \Delta t$.

For the other variables

$$\begin{aligned}
(v_x)_{i,j}^{n+1} &= (v_x)_{i,j}^{n} + \left(\frac{\partial v_x}{\partial t}\right)_{\text{av}} \Delta t \\
(v_y)_{i,j}^{n+1} &= (v_y)_{i,j}^{n} + \left(\frac{\partial v_y}{\partial t}\right)_{\text{av}} \Delta t \\
e_{i,j}^{n+1} &= e_{i,j}^{n} + \left(\frac{\partial e}{\partial t}\right)_{\text{av}} \Delta t
\end{aligned}$$

Fluid equations

The $\left(\frac{\partial\rho}{\partial t}\right)_{\text{av}}$ is computed using a predictor-corrector methodology.

Predictor

Using the continuity equation (forward differences)

$$\left(\frac{\partial\rho}{\partial t}\right)^n_{i,j} = -\left(\rho^n_{i,j}\frac{(v_x)^n_{i+1,j}-(v_x)^n_{i,j}}{\Delta x}+(v_x)^n_{i,j}\,\frac{\rho^n_{i+1,j}-\rho^n_{i,j}}{\Delta x}\right.$$
$$\left.+\,\rho^n_{i,j}\frac{(v_y)^n_{i,j+1}-(v_y)^n_{i,j}}{\Delta y}+(v_y)^n_{i,j}\,\frac{\rho^n_{i,j+1}-\rho^n_{i,j}}{\Delta y}\right)$$

The predicted value is

$$\boxed{(\bar{\rho})^{n+1}_{i,j}=\rho^n_{i,j}+\left(\frac{\partial\rho}{\partial t}\right)^n_{i,j}\Delta t}$$

A similar procedure is used for the other variables

Corrector

Using the continuity equation (backward differences)

$$\left(\frac{\partial \bar{\rho}}{\partial t}\right)_{i,j}^{n+1} = -\left(\bar{\rho}_{i,j}^{n}\frac{(\bar{v}_x)_{i,j}^{n+1}-(\bar{v}_x)_{i-1,j}^{n+1}}{\Delta x} + (\bar{v}_x)_{i,j}^{n+1}\frac{\bar{\rho}_{i,j}^{n}-\bar{\rho}_{i-1,j}^{n}}{\Delta x}\right.$$
$$\left. + \bar{\rho}_{i,j}^{n+1}\frac{(\bar{v}_y)_{i,j}^{n}-(\bar{v}_y)_{i,j-1}^{n}}{\Delta y} + (\bar{v}_y)_{i,j}^{n+1}\frac{\bar{\rho}_{i,j+1}^{n}-\bar{\rho}_{i,j}^{n}}{\Delta y}\right)$$

The average value of the derivative

$$\boxed{\left(\frac{\partial \rho}{\partial t}\right)_{\text{av}} = \frac{1}{2}\left(\left(\frac{\partial \rho}{\partial t}\right)_{i,j}^{n} + \left(\frac{\partial \bar{\rho}}{\partial t}\right)_{i,j}^{n+1}\right)}$$

The scheme

$$\boxed{\rho_{i,j}^{n+1} = \rho_{i,j}^{n} + \left(\frac{\partial \rho}{\partial t}\right)_{\text{av}}\Delta t}$$

Fluid equations

The fluid equations can be expressed in conservative form

$$\frac{\partial U}{\partial t} + \frac{\partial F}{\partial x} + \frac{\partial G}{\partial y} = J$$

where

$$U = \begin{pmatrix} \rho \\ \rho v_x \\ \rho v_y \\ \rho\left(e + \frac{V^2}{2}\right) \end{pmatrix}, \quad F = \begin{pmatrix} \rho v_x \\ \rho\left(v_x\right)^2 + p \\ \rho v_x v_y \\ \rho\left(e + \frac{V^2}{2}\right) v_x + p v_x \end{pmatrix},$$

$$G = \begin{pmatrix} \rho v_y \\ \rho v_x v_y \\ \rho\left(v_y\right)^2 + p \\ \rho\left(e + \frac{V^2}{2}\right) v_y + p v_y \end{pmatrix}, \quad J = \begin{pmatrix} 0 \\ \rho f_x \\ \rho f_y \\ \rho\left(v_x f_x + v_y f_y\right) \end{pmatrix},$$

Fluid equations

- MacCormack's method can be applied to the conservative formulation, but the physical variables have to be isolated from the components of U in each time step.
- This method can present oscillations in certain conditions. To stabilize the method, an artificial viscosity term can be added

$$S_{i,j}^n = \frac{C_x \left|p_{i-1,j} - 2p_{i,j} + p_{i+1,j}\right|}{p_{i-1,j} + 2p_{i,j} + p_{i+1,j}} \left(U_{i-1,j}^n - 2U_{i,j}^n + U_{i+1,j}^n\right) \\ \frac{C_y \left|p_{i,j-1} - 2p_{i,j} + p_{i,j+1}\right|}{p_{i,j-1} + 2p_{i,j} + p_{i,j+1}} \left(U_{i,j-1}^n - 2U_{i,j}^n + U_{i,j+1}^n\right)$$

which is a fourth-order term. The parameters C_x and C_y range from 0.01 to 0.3

Fluid equations

The viscosity term is applied in two steps

$$\bar{U}_{i,j}^{n+1} = U_{i,j}^{n} + \left(\frac{\partial U}{\partial t}\right)_{i,j}^{n} \Delta t + S_{i,j}^{n}$$

$$U_{i,j}^{n+1} = U_{i,j}^{n} + \left(\frac{\partial U}{\partial t}\right)_{\mathrm{av}}^{n} \Delta t + \bar{S}_{i,j}^{n+1}$$

Incompressible viscous flow: The pressure correction

We consider the incompressible Navier-Stokes equations

$$
\begin{aligned}
\vec{\nabla}\vec{v} &= 0 \\
\rho\frac{Dv_x}{Dt} &= -\frac{\partial p}{\partial x} + \mu\nabla^2 v_x + \rho f_x \\
\rho\frac{Dv_y}{Dt} &= -\frac{\partial p}{\partial y} + \mu\nabla^2 v_y + \rho f_y
\end{aligned}
$$

If we apply MacCormack's technique, the time step is restricted by stability conditions. An approximate stability condition is

$$\Delta t \leq \frac{1}{|v_x|/\Delta x + |v_y|/\Delta y + a\sqrt{1/(\Delta x)^2 + 1/(\Delta y)^2}}$$

where a is the speed of sound

Incompressible viscous flow: The pressure correction

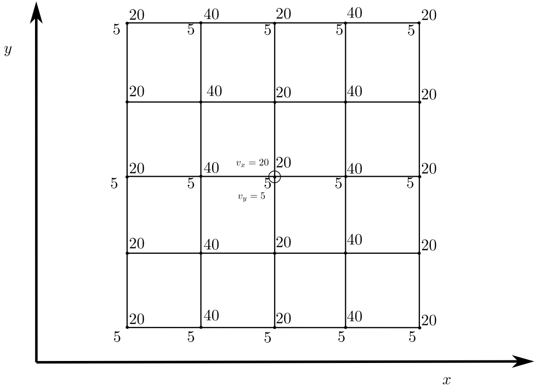
- For a compressible flow the speed of sound is finite.
- For incompressible flow the speed of sound is theoretically infinite and the stability condition yields to $\Delta t = 0$.
- CFD solutions for incompressible Navier-Stokes equations are different from those used for the compressible Navier-Stokes.
- The continuity equation for incompressible flow is

$$\frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} = 0$$

a central difference scheme

$$\frac{(v_x)_{i+1,j} - (v_x)_{i-1,j}}{2\Delta x} + \frac{(v_y)_{i,j+1} - (v_y)_{i,j-1}}{2\Delta y} = 0$$

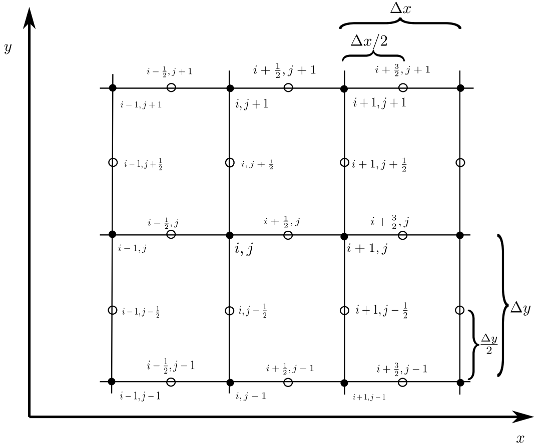
Incompressible viscous flow: The pressure correction



Incompressible viscous flow: The pressure correction

- The difference equation numerically allows the chequerboard velocity distribution given in the Figure.
- In the x direction, v_x varies as 20, 40, 20, 40, etc., at successive grid points, and in the y direction, v_y varies as 5, 2, 5, 2, etc., at successive grid points.
- The chequerboard velocity distribution is basically nonsense in terms of any real, physical flow field. A similar behaviour is found from the pressure is central schemes are used for the derivatives.
- Given the weakness of the central difference formulation described above, we should justifiably feel uncomfortable, and we should look for some "fix" before embarking on the solution of a given problem.

As a solution a staggered grid is proposed



Incompressible viscous flow: The pressure correction

- The pressures and velocities are calculated at different points.
- When $(v_x)_{i+1/2,j}$ is calculated a central difference is used for

$$\left(\frac{\partial p}{\partial x}\right)_{i,j} \approx \frac{p_{i+1,j} - p_{i,j}}{\Delta x}$$

The pressure correction

The pressure correction is an iterative procedure (SIMPLE method)

1. An initial guess is used for the pressures $p^*_{i,j}$.
2. With these values of p^* the values of v_x and v_y are computed from the momentum equations.
3. Using the continuity equation a pressure correction p' is obtained,

$$p = p^* + p'$$

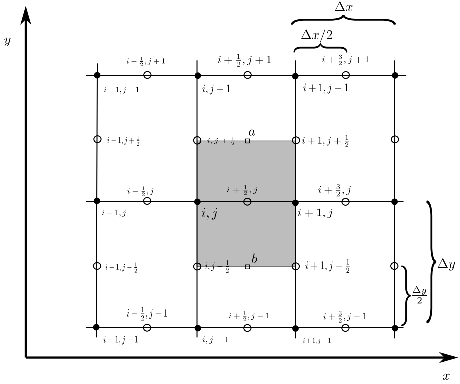
With p', correction for the velocities $(v_x)'$, $(v_y)'$,

$$v_x = v_x^* + v_x' \;, \quad v_y = v_y^* + v_y' \;,$$

with the new value of p return to step 2.

The pressure correction

Using the computational cell



$$\bar{v}_y = \frac{1}{2}\left((v_y)_{i,j+1/2} + (v_y)_{i+1,j+1/2}\right)$$
$$v_y = \frac{1}{2}\left((v_y)_{i,j-1/2} + (v_y)_{i+1,j-1/2}\right)$$

The pressure correction

The momentum equation centred at $\left(i+\frac{1}{2},j\right)$

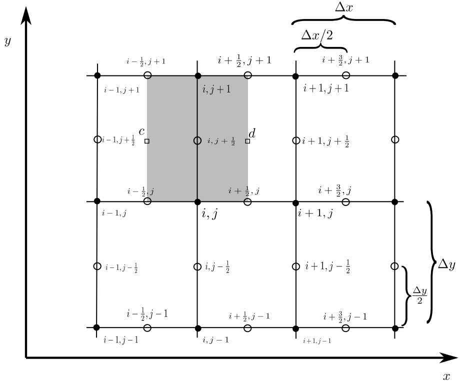
$$(\rho v_x)^{n+1}_{i+1/2,j} = (\rho v_x)^{n}_{i+1/2,j} + A\Delta t - \frac{\Delta t}{\Delta x}\left(p^n_{i+1,j} - p^n_{i,j}\right)$$

where

$$\begin{aligned} A &= -\left(\frac{\left(\rho v_x^2\right)^n_{i+3/2,j} - \left(\rho v_x^2\right)^n_{i-1/2,j}}{2\Delta x} + \frac{(\rho v_x \bar{v}_y)_{i+1/2,j+1} - (\rho v_x v_y)_{i+1/2,j-1}}{2\Delta y}\right) \\ & +\mu\left(\frac{(v_x)^n_{i+3/2,j} - 2\,(v_x)^n_{i+1/2,j} + (v_x)^n_{i-1/2,j}}{\Delta x^2}\right. \\ & \left. + \frac{(v_x)^n_{i+1/2,j+1} - 2\,(v_x)^n_{i+1/2,j} + (v_x)^n_{i+1/2,j-1}}{\Delta y^2}\right) \end{aligned}$$

The pressure correction

Now using the computational cell



$$
\begin{aligned}
v_x &= \frac{1}{2}\left((v_x)_{i-1/2,j} + (v_x)_{i-1/2,j+1}\right) \\
\bar{v}_x &= \frac{1}{2}\left((v_y)_{i+1/2,j} + (v_x)_{i+1/2,j+1}\right)
\end{aligned}
$$

The pressure correction

The momentum equation centred at $\left(i, j+\frac{1}{2}\right)$

$$(\rho v_y)^{n+1}_{i,j+1/2} = (\rho v_y)^{n}_{i,j+1/2} + B\Delta t - \frac{\Delta t}{\Delta y}\left(p^n_{i,j+1} - p^n_{i,j}\right)$$

where

$$\begin{aligned} B &= -\left(\frac{(\rho v_y \bar{v}_x)^n_{i+1,j+1/2} - (\rho v_x v_y)^n_{i-1,j+1/2}}{2\Delta x} + \frac{\left(\rho v_y^2\right)_{i,j+3/2} - \left(\rho v_y^2\right)_{i,j-1/2}}{2\Delta y}\right) \\ &\quad +\mu\left(\frac{(v_y)^n_{i+1,j+1/2} - 2\,(v_y)^n_{i,j+1/2} + (v_y)^n_{i-1,j+1/2}}{\Delta x^2}\right. \\ &\quad \left. + \frac{(v_y)^n_{i,j+3/2} - 2\,(v_x)^n_{i,j+1/2} + (v_x)^n_{i,j-1/2}}{\Delta y^2}\right) \end{aligned}$$

The pressure correction

The iteration begins with p^* and

$$(\rho v_x^*)_{i+1/2,j}^{n+1} = (\rho v_x^*)_{i+1/2,j}^{n} + A^* \Delta t - \frac{\Delta t}{\Delta x}\left(p_{i+1,j}^{*n} - p_{i,j}^{*n}\right)$$

$$\left(\rho v_y^*\right)_{i,j+1/2}^{n+1} = \left(\rho v_y^*\right)_{i,j+1/2}^{n} + B^* \Delta t - \frac{\Delta t}{\Delta y}\left(p_{i,j+1}^{*n} - p_{i,j}^{*n}\right)$$

A correction for the velocities is used

$$(\rho v_x')_{i+1/2,j}^{n+1} = (\rho v_x^*)_{i+1/2,j}^{n} - \frac{\Delta t}{\Delta x}\left(p'_{i+1,j} - p'_{i,j}\right)$$

$$\left(\rho v_y'\right)_{i,j+1/2}^{n+1} = \left(\rho v_y^*\right)_{i,j+1/2}^{n} - \frac{\Delta t}{\Delta y}\left(p'_{i,j+1} - p'_{i,j}\right)$$

The pressure correction

Using the continuity equation

$$\frac{(\rho v_x)_{i+1/2,j} - (\rho v_x)_{i-1/2,j}}{\Delta x} + \frac{(\rho v_y)_{i,j+1/2} - (\rho v_y)_{i,j-1/2}}{\Delta y} = 0$$

we obtain

$$\boxed{a p'_{i,j} + b p'_{i+1,j} + b p'_{i-1,j} + c p'_{i,j+1} + c p'_{i,j-1} + d = 0}$$

where

$$a = 2\left(\frac{\Delta t}{\Delta x^2} + \frac{\Delta t}{\Delta y^2}\right) \ , \quad b = -\frac{\Delta t}{\Delta x^2}, \quad c = -\frac{\Delta t}{\Delta y^2},$$

$$d = \frac{1}{\Delta x}\left((\rho v_x^*)_{i+1/2,j} - (\rho v_x^*)_{i-1/2,j}\right) + \frac{1}{\Delta y}\left(\left(\rho v_y^*\right)_{i,j+1/2} - \left(\rho v_y^*\right)_{i,j-1/2}\right)$$

The pressure correction